Critical Phenomena in Gravitational Collapse

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Universality and Scaling in Gravitational Collapse of a Massless Scalar Field

Matthew W. Choptuik
Center for Relativity, University of Texas at Austin, Austin, Texas 78712-1081
(Received 22 September 1992)

I summarize results from a numerical study of spherically symmetric collapse of a massless scalar field. I consider families of solutions, $\mathcal{S}[p]$, with the property that a critical parameter value, $p^\star$, separates solutions containing black holes from those which do not. I present evidence in support of conjectures that (1) the strong-field evolution in the $p \to p^\star$ limit is universal and generates structure on arbitrarily small spatiotemporal scales and (2) the masses of black holes which form satisfy a power law $M_{\rm BH} \propto |p - p^\star|^\gamma$, where $\gamma \approx 0.37$ is a universal exponent.

Outline

- A numerical experiment
- Critical phenomena: uniqueness, self-similarity, and scaling
- Recent results for critical collapse of gravitational waves

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

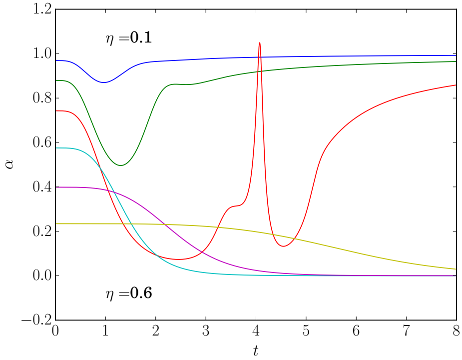
- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different amplitudes η...
- Have *critical parameter* η_* so that

$$\eta < \eta_* \qquad \alpha \to 1 \qquad \to \text{flat space}$$

$$\eta > \eta_* \qquad \alpha \to 0 \qquad \to \text{black hole}$$



$$0.3 < \eta_* < 0.4$$

A numerical experiment...

- Consider massless scalar field

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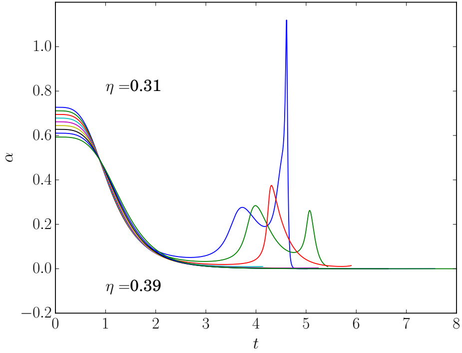

$$0.30 < \eta_* < 0.31$$

A numerical experiment...

- Consider massless scalar field

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coupled to Einstein's equations

- Initial data

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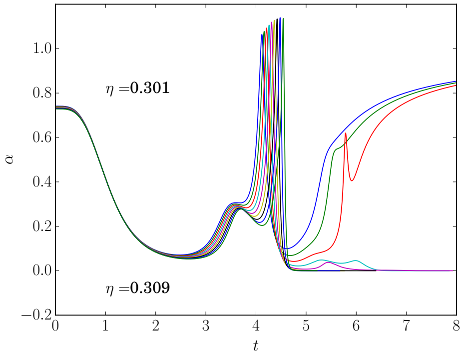

$$0.303 < \eta_* < 0.304$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

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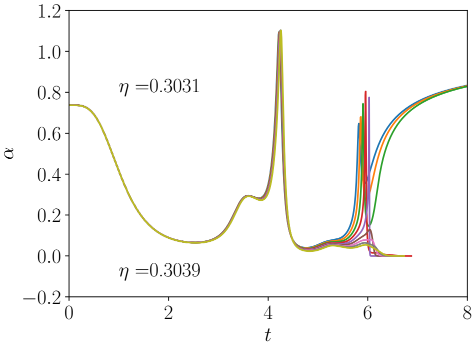

$$0.3033 < \eta_* < 0.3034$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different amplitudes η...
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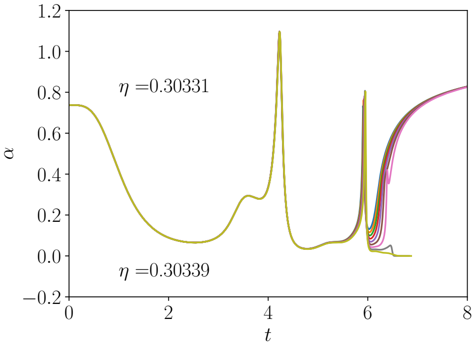

$$0.30337 < \eta_* < 0.30337$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta \exp(-R^2/R_0^2)$$

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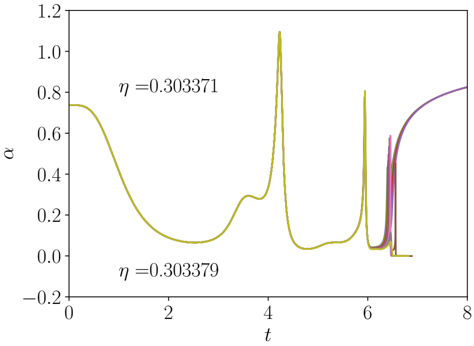


$$0.303375 < \eta_* < 0.303376$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

coupled to Einstein's equations

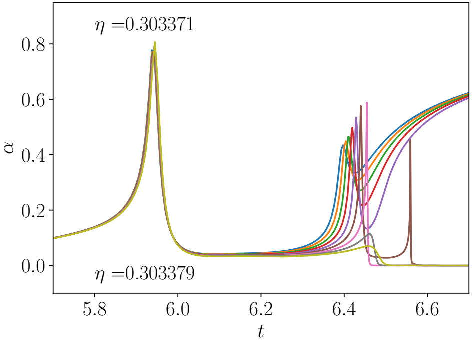
- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different amplitudes η...
- Have *critical parameter* η_* so that

$\eta < \eta_*$ $\alpha \to 1$ $\to$ flat space

$\eta > \eta_*$ $\alpha \to 0$ $\to$ black hole



$$0.303375 < \eta_* < 0.303376$$

A numerical experiment...

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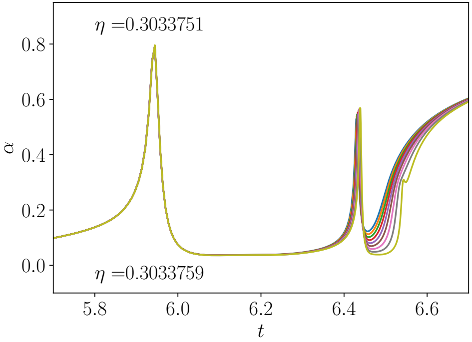

$$0.3033759 < \eta_* < 0.3033760$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

coupled to Einstein's equations

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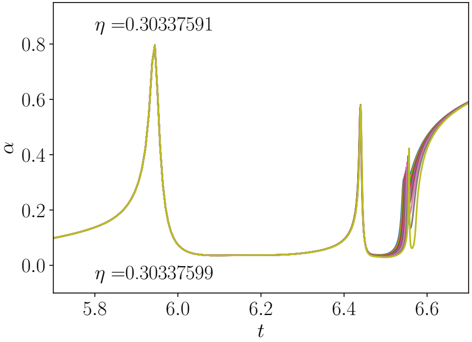


$$0.30337599 < \eta_* < 0.30337600$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

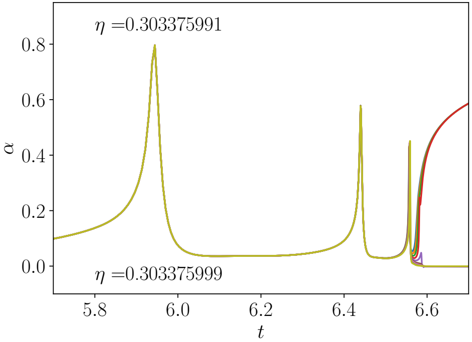
coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different amplitudes η...
- Have *critical parameter* η_* so that

$\eta < \eta_*$	$\alpha \to 1$	$\to$ flat space
$\eta > \eta_*$	$\alpha \to 0$	$\to$ black hole



$$0.303375994 < \eta_* < 0.303375995$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different amplitudes η...
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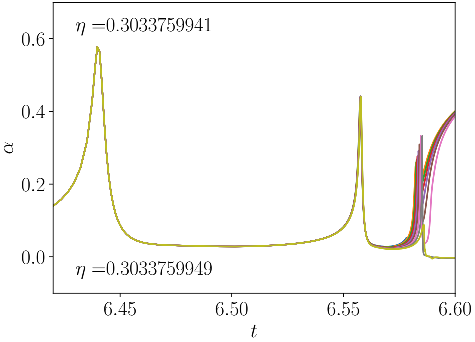

$$0.3033759947 < \eta_* < 0.3033759948$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different amplitudes η...
- Have *critical parameter* η_* so that

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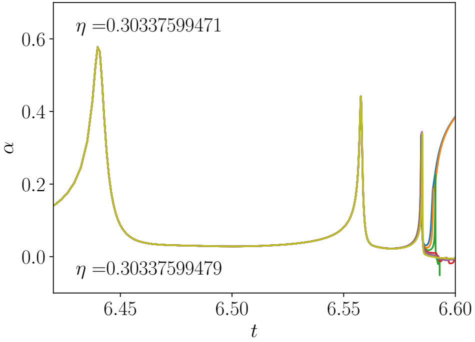

$$0.30337599472 < \eta_* < 0.30337599473$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta \exp(-R^2/R_0^2)$$

- try out different amplitudes η...
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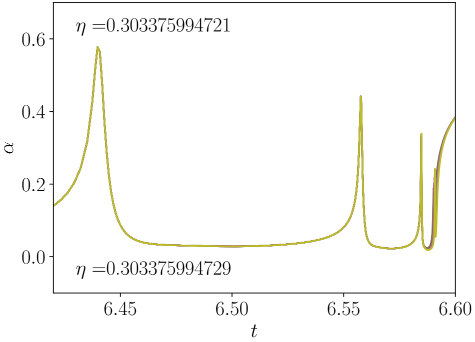

$$0.303375994729 < \eta_* < 0.303375994730$$

A numerical experiment...

- Consider massless scalar field

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

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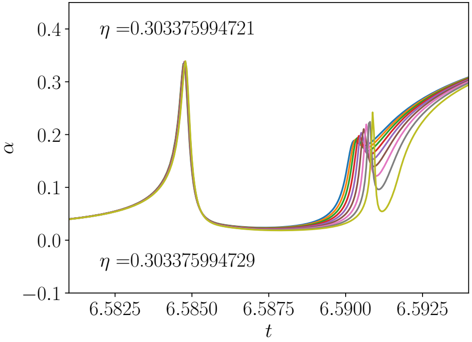
$$\eta > \eta_* \qquad \alpha \to 0 \qquad \to \text{black hole}$$



$$0.303375994729 < \eta_* < 0.303375994730$$

A numerical experiment...

- Consider massless scalar field

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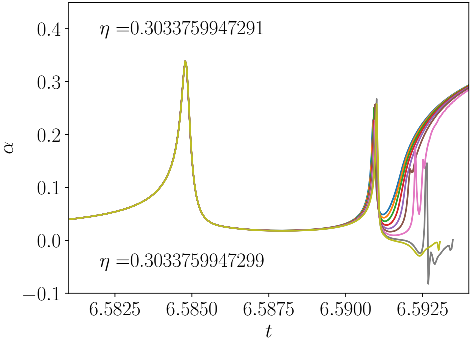
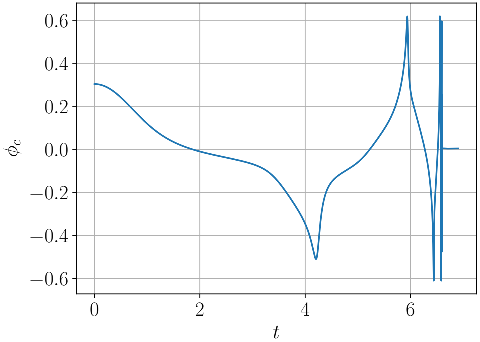


$$0.3033759947297 < \eta_* < 0.3033759947298$$

Critical Solution

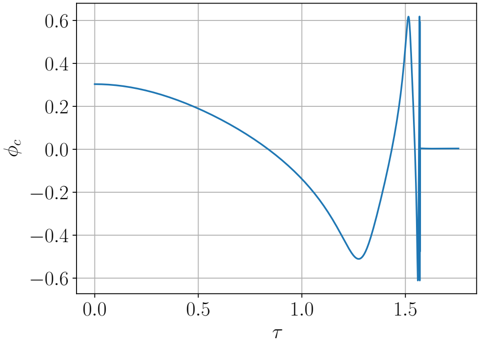
- Look at scalar field ϕ for $\eta \to \eta_*$ at $r = 0$



Critical Solution

- Look at scalar field ϕ for $\eta \to \eta_*$ at $r = 0$
- plot as function of proper time τ
- oscillations accumulate at *accumulation time*

$$\tau_* \simeq 1.5698$$



Critical Solution

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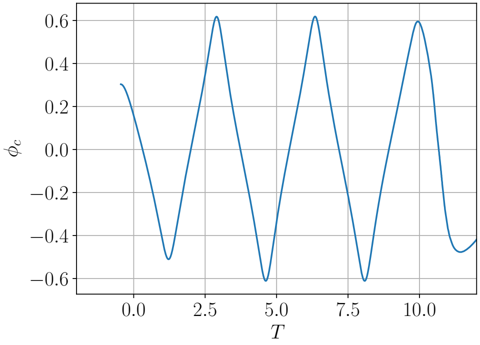
- oscillations accumulate at *accumulation time*

$$\tau_* \simeq 1.5698$$

- plot as function of *slow time*

$$T \equiv -\log(\tau_* - \tau)$$



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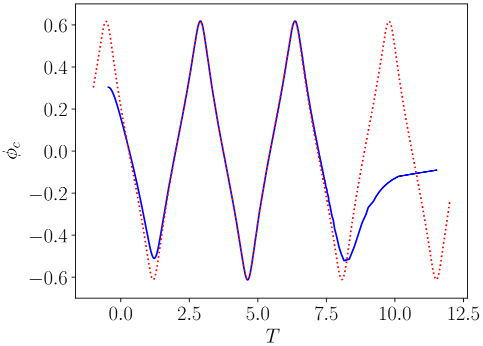


Critical solution performs periodic oscillations in slow time T: *discrete self-similarity*

Can we form arbitrarily small black holes?

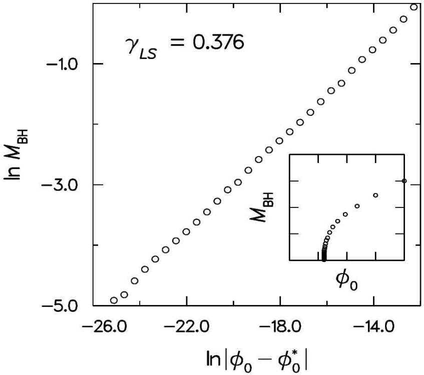
Plot mass M of forming black hole as function of parameter η

- find power-law scaling

$$M \simeq (\eta - \eta_*)^\gamma$$

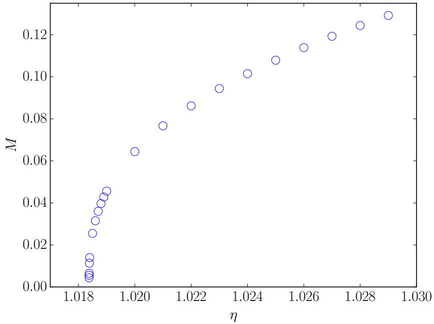
  with *critical exponent* $\gamma \simeq 0.37$

- Looks familiar??



[Choptuik, 1998]

Looks familiar?



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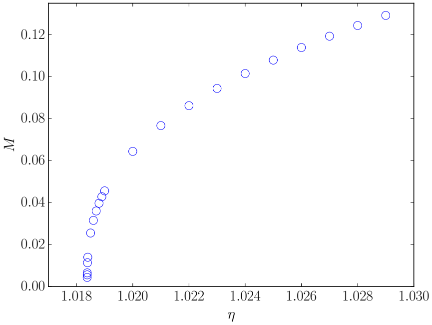
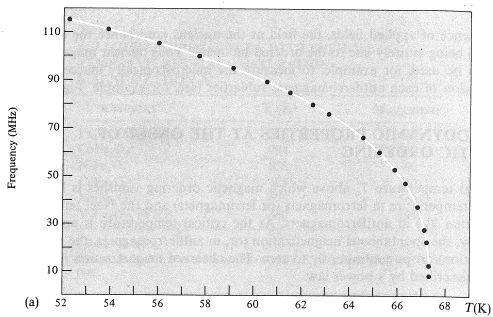




Critical Phenomena

in the absence of applied fields, the field at the nucleus (and hence the resonance frequency) being entirely due to the ordered moments. Thus nuclear magnetic resonance can be used, for example, to measure the macroscopically inaccessible net magnetization of each antiferromagnetic sublattice (see, for example, Figure 33.4).

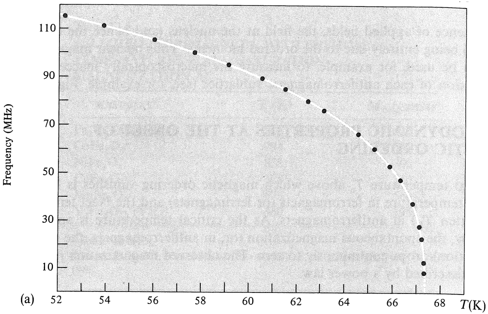
THERMODYNAMIC PROPERTIES AT THE ONSET OF MAGNETIC ORDERING

The critical temperature T_c above which magnetic ordering vanishes is known as the Curie temperature in ferromagnets (or ferrimagnets) and the Néel temperature (often written T_N) in antiferromagnets. As the critical temperature is approached from below, the spontaneous magnetization (or, in antiferromagnets, the sublattice magnetization) drops continuously to zero. The observed magnetization just below T_c is well described by a power law.

$$M(T) \sim (T_c - T)^\beta, \quad (33.1)$$

where β is typically between 0.33 and 0.37 (see Figure 33.4).

The onset of ordering is also signaled as the temperature drops to T_c from above,



Magnetic field M in MnF_2 as function of temperature T
[Ashcroft & Mermin, *Solid State Physics*, 1976]

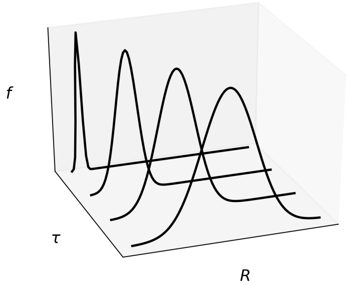
Critical Phenomena in Gravitational Collapse

- Consider matter model (e.g. scalar field, fluid, vacuum...)
- Consider family of initial data parametrized by η and evolve...
- Critical parameter η_* separates:
  - *supercritical* data: form black hole
  - *subcritical* data: don't
- in vicinity of η_* observe *critical phenomena*:
  - dimensional quantities display *scaling*, e.g.

    $$M \simeq (\eta - \eta_*)^\gamma$$

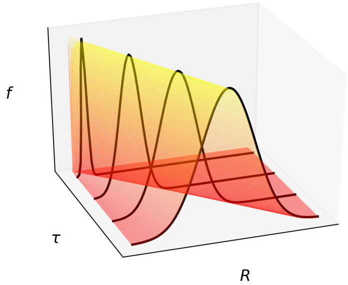
    with *critical exponent* γ: depends on matter model, but not on parametrization of initial data
  - spacetime approaches unique *self-similar solution*

Self-similarity



- Solution contracts without changing shape. . .

Self-similarity



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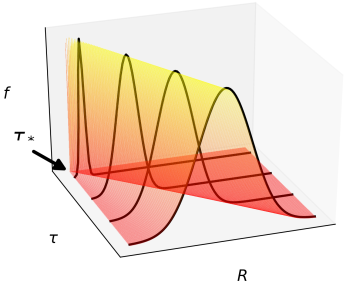
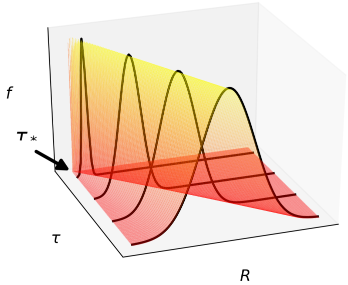
Self-similarity



- Solution contracts without changing shape. . .
- . . . towards accumulation event at $\tau = \tau_*$

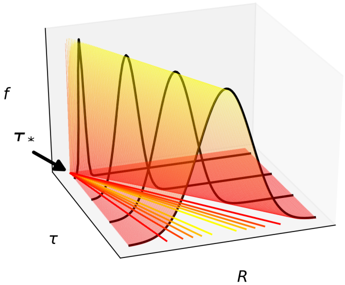
Self-similarity



- Solution contracts without changing shape...
- ...towards accumulation event at $\tau = \tau_*$
- radius R proportional to $\tau_* - \tau$, so define *self-similar radius*

$$\xi \equiv \frac{R}{\tau_* - \tau}$$

Self-similarity



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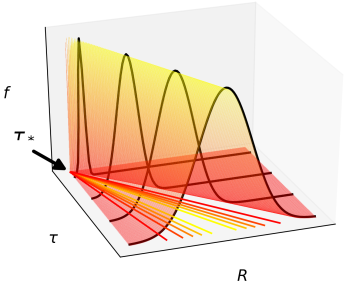
- ...towards accumulation event at $\tau = \tau_*$
- radius R proportional to $\tau_* - \tau$, so define *self-similar radius*

$$\xi \equiv \frac{R}{\tau_* - \tau}$$

- dimensionless quantities are functions of ξ only,

$$Z = Z_*(\xi)$$

Self-similarity



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- radius R proportional to $\tau_* - \tau$, so define *self-similar radius*

$$\xi \equiv \frac{R}{\tau_* - \tau}$$

- dimensionless quantities are functions of ξ only,

$$Z = Z_*(\xi)$$

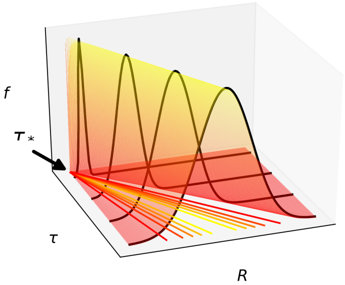
No preferred length scale, so what sets scale of forming black hole, say??

Three phases of evolution

- Phase I:
  from initial data to something close to critical solution
  (how close? depends on degree of fine-tuning. . .)
- Phase II:
  critical solution plus perturbation
  (until perturbation becomes nonlinear)
- Phase III:
  dispersion or collapse to black hole

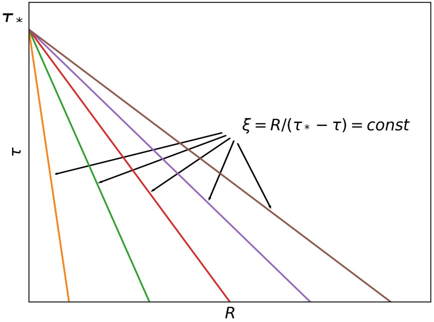
Length scales set by size of the self-similar solution at transition from Phase II to III

Phase II: Perturbations of Critical Solution



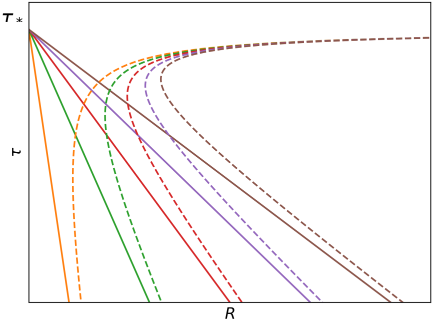
- Consider perturbations ζ of the critical solution

Phase II: Perturbations of Critical Solution



- Consider perturbations ζ of the critical solution

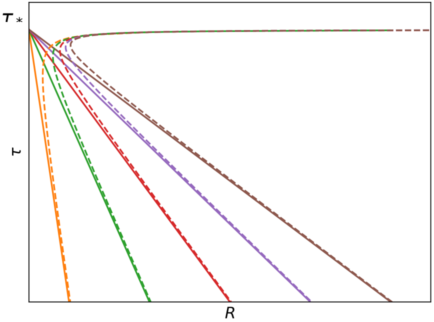
Phase II: Perturbations of Critical Solution



- Consider perturbations ζ of the critical solution
- assume that only one mode is unstable
- grows at rate γ in $T = -\log(\tau_* - \tau)$

$$\zeta \propto e^{\lambda T} = (\tau_* - \tau)^{-\lambda}$$

Phase II: Perturbations of Critical Solution



- Consider perturbations ζ of the critical solution
- assume that only one mode is unstable
- grows at rate γ in $T = -\log(\tau_* - \tau)$

$$\zeta \propto e^{\lambda T} = (\tau_* - \tau)^{-\lambda}$$

- to leading order also proportional to $\eta - \eta_*$,

$$\zeta \propto (\eta - \eta_*)(\tau_* - \tau)^{-\lambda}$$

Phase II: Perturbations of Critical Solution

Recall:

$$\zeta \propto (\eta - \eta_*)(\tau_* - \tau)^{-\lambda}$$

Now...

- mode becomes nonlinear when $\zeta = \bar{\zeta}$, say
- length-scale R at moment when ζ reaches $\bar{\zeta}$ given by

$$R \propto (\tau_* - \tau) \propto \bar{\zeta}^{-1/\lambda}(\eta - \eta_*)^{1/\lambda}$$

- scaling laws, e.g.

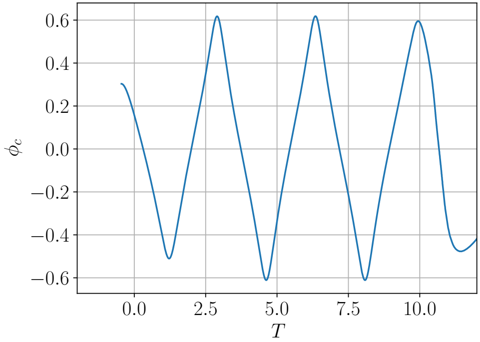
$$M \propto (\eta - \eta_*)^{\gamma}$$

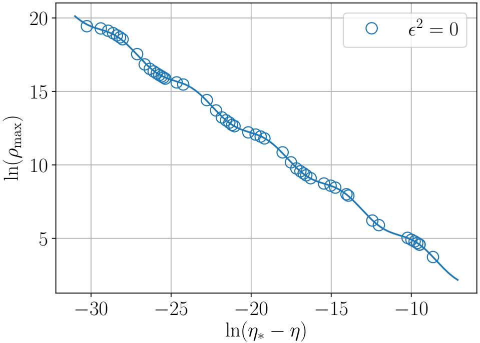
  with $\gamma = 1/\lambda$
  [Koike *et.al.*, 1995; Maison, 1995]

Continuous versus discrete self-similarity

Self-similarity can be...

- *continuous* (CSS) (e.g. fluid)
- *discrete* (DSS) (e.g. scalar fields: expect "super-imposed" oscillations)





Key ingredients of critical collapse

- Unique critical solution, either CSS or DSS
- Single unstable mode, Lyapunov exponent λ
- Power-law scaling with critical exponent $\gamma = 1/\lambda$
- Pretty well established in spherical symmetry…

…but what about non-spherical cases??

Critical collapse of gravitational waves

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Critical Behavior and Scaling in Vacuum Axisymmetric Gravitational Collapse

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We report a second example of critical behavior in gravitational collapse. Collapse of axisymmetric gravitational wave packets is computed numerically for a one-parameter family of initial data. A black hole first appears along the sequence at a critical parameter value $p^\star$. As with spherical scalar-field collapse, a power law is found to relate black-hole mass (the order parameter) and critical separation: $M_{\rm BH} \propto |p - p^\star|^\beta$. The critical exponent is $\beta \simeq 0.37$, remarkably close to that observed by Choptuik. Near-critical evolutions produce echoes from the strong-field region which appear to exhibit scaling.

Numerous attempts to reproduce this...

Despite many attempts...

[Alcubierre *et.al.*, 2000; Garfinkle & Duncan, 2001; Santamaria, 2006; Rinne, 2008; Sorkin, 2011; Hilditch *et.al.*, 2013]

... real progress in reproducing results of Abrahams & Evans only recently:

[Hilditch *et.al.*, 2017; Ledvinka & Khirnov, 2021; Fernández *et.al.*, 2022]

... will complement with new results...

Initial data

Typically, axisymmetric initial data describing gravitational waves are set up in one of two ways:

- *Brill waves*:
  deform conformally related metric with seed function, solve linear (flat) elliptic equation for conformal factor
  [Brill, 1959]
- *Teukolsky waves*:
  start with analytical wave-like solution to linearized Einstein equations, then "dress up" to satisfy constraints
  [Teukolsky, 1982; Rinne, 2008]

Evolution

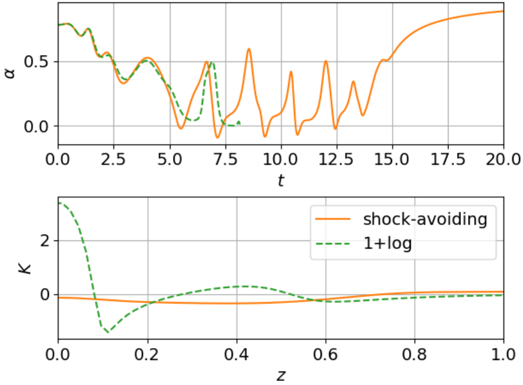
Many previous attempts used 1+log slicing,

$$(\partial_t - \beta^i \partial_i)\alpha = -\alpha^2 f(\alpha) K$$

with

$$f(\alpha) = 2/\alpha$$

Very successful in many cases, but can lead to coordinate shocks...
[Alcubierre, 1997; 2003]



[TWB *et.al.*, in prep.]

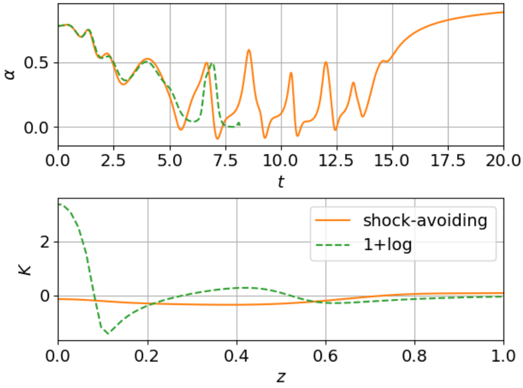
Evolution

Consider alternatives...

- approximate maximal slicing in BSSN formalism
  [Ledvinka & Khirnov, 2018]
- gauge-source functions in generalized harmonic formalism
  [Hilditch *et.al.*, 2017]
- *shock-avoiding* slicing condition in BSSN: use

$$f(\alpha) = 1 + \kappa/\alpha^2$$

[Alcubierre, 1997; TWB & Hilditch, 2022]



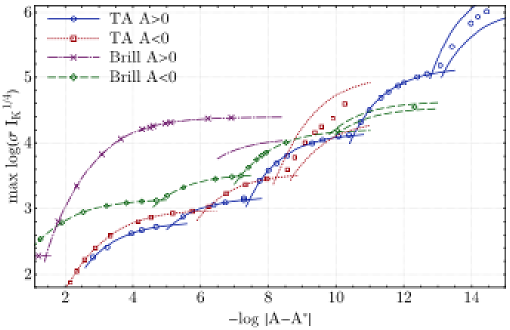
[TWB *et.al.*, in prep.]

Uniqueness...

- Pick different families of initial data parametrized by amplitude A
- Fine-tune to black-hole threshold A_*
- Plot maximum attained curvature I for subcritical data
- If critical solution were unique and DSS would expect power law

$$I \simeq |A - A_*|^\gamma$$

plus periodic wiggles with unique γ

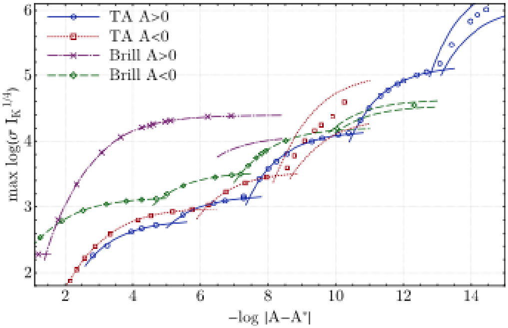


[Ledvinka & Khirnov, 2021]

Uniqueness...

Unlike in spherically symmetric case...

- γ depends on family
- critical solution family-dependent
- No clear evidence for "threshold" solutions being DSS



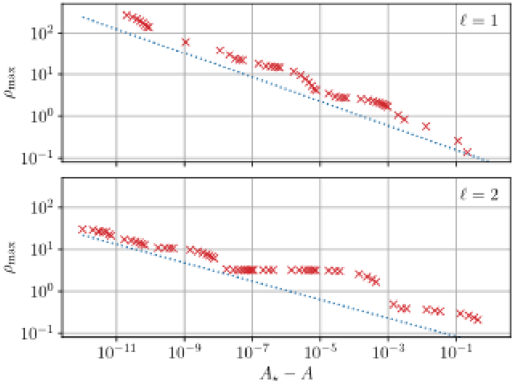
[Ledvinka & Khirnov, 2021]

Uniqueness...

Compare with gravitational collapse of electromagnetic waves...

- distinct threshold solutions for dipole and quadrupole waves
- each one only approximately DSS
- results in distinct values of γ

[TWB *et.al.*, 2019; Perez Mendoza & TWB, 2021]



Uniqueness

In absence of spherical symmetry...

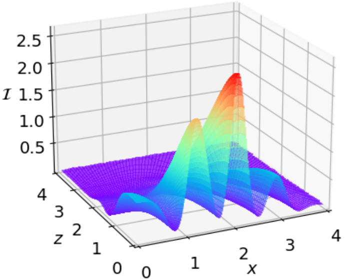
- ... should not expect existence of unique threshold solution
- ... may have multiple centers of collapse
- ... any accumulation event may not be at center of symmetry

But: do gravitational-wave families with DSS threshold solutions exist??

Self-similarity

- Consider superposition of ingoing and outgoing Teukolsky waves
- Use BSSN code in spherical polar coordinates
- evolve with shock-avoiding slicing condition
- analyze Weyl scalars $\mathcal{I}$ and $\mathcal{J}$
- plot in terms of self-similar coordinates

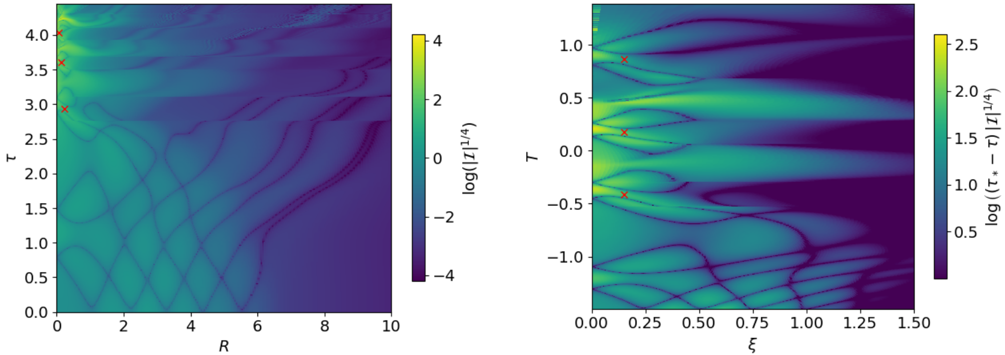
$$T = -\log(\tau_* - \tau) \qquad \xi = R/(\tau_* - \tau)$$



[TWB *et.al.*, in prep]

Self-similarity

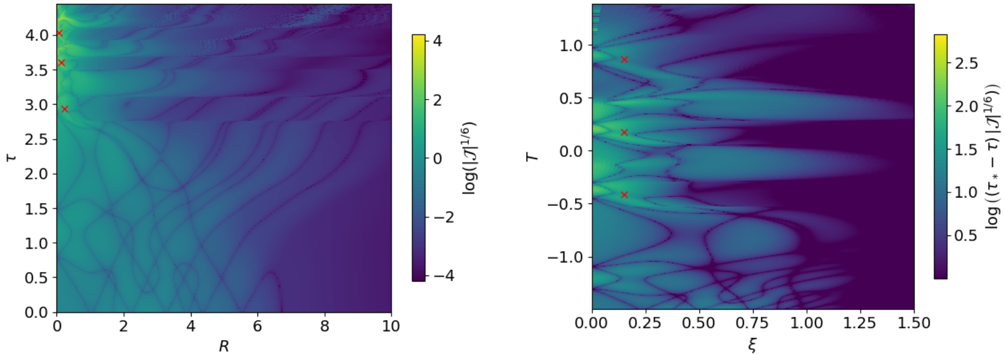
Weyl scalar $\mathcal{I}$ for near-critical solution $(A \simeq A_*)$ in equatorial plane



Patterns repeat with period of approximately $\Delta \simeq 0.65$

Self-similarity

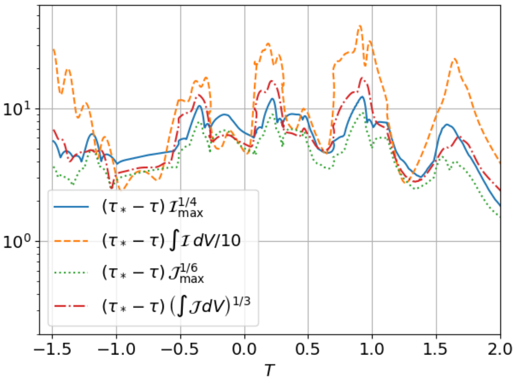
Weyl scalar $\mathcal{J}$ for near-critical solution $(A \simeq A_*)$ in equatorial plane



Patterns repeat with period of approximately $\Delta \simeq 0.65$

Self-similarity

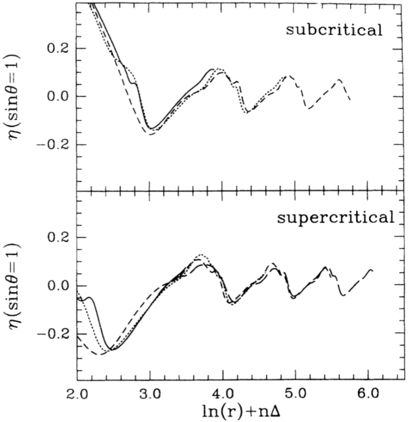
- Plot curvature measures as function of time T for near-critical solution
- *approximately* periodic with period $\Delta \simeq 0.65$



Self-similarity

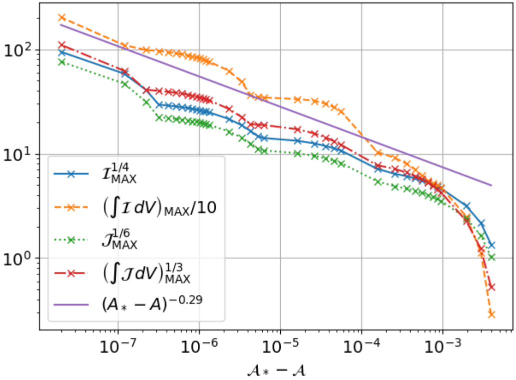
- Strong evidence of at least *approximate* DSS
- but is DSS exact?
- Not clear...

[Abrahams & Evans, 1994]



Scaling

- Plot maximum curvature attained as function of $A_* - A$
- approximate power law with $\gamma \simeq 0.29$



Summary

- Critical phenomena in gravitational collapse
- Uniqueness, self-similarity, and scaling
- Well understood in spherical symmetry
- Progress for vacuum gravitational waves
  - likely no unique critical solution
  - but there exist at least approximate DSS threshold solutions