# Rotating stellar systems and their black holes

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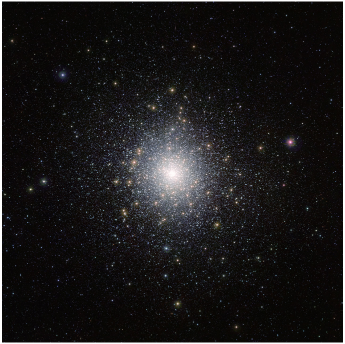
*with Sam Bonsor and Jacques Vanneste*



Star cluster
(47 Tucanae)

Galaxy
(Andromeda)

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1, i\neq j}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

"Gravitational N-body problem"

# Relaxation processes in self-gravitating systems

- **Dynamical time:** time scale needed by a test star to cross the stellar system

$$t_D = 2r/\sigma$$

- **Two-star relaxation time:** time scale beyond which the cumulative effect on $v_\perp$ due to subsequent two-star gravitational encounters becomes comparable to the starting kinetic energy of a test star

$$t_{rel} \approx \frac{v_{start}^3}{8\pi G^2 n(r) m^2 \ln \Lambda}$$

## Quasi-relaxed systems

e.g. star clusters $N \approx 10^4 \div 10^6$

$$t_D << t_{rel} < t_{age}$$

$$10^5 yr << 10^8 yr < 10^{10} yr$$

Two-stars relaxation processes should have had enough time to bring them close to a thermodynamically relaxed state, with their distribution function close to a Maxwellian.

## Partially relaxed systems

e.g. galaxies $N \approx 10^{10} \div 10^{11}$

$$t_D < t_{age} << t_{rel}$$

$$10^8 yr < 10^{10} yr << 10^{18} yr$$

Unable to complete relaxation during an early phase of rapid evolution (“violent relaxation”); should be thought of as truly collisionless stellar systems.

# VOCABULARY

- Kinetic description
  Distribution functions in phase space $f = f(\mathbf{x}, \mathbf{v})$
  Self-consistent equilibrium solution of the Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\partial \Phi}{\partial \mathbf{x}} = 0$$

$$\nabla^2 \Phi = 4\pi G \int f d^3 \boldsymbol{v}$$

*Phase space dependence only via isolating integrals of motion ("Jeans Theorem").*

- Fluid description
  Density-Potential pairs $(\rho, \Phi)$
  Solution of the moments equations, with physical closure ("equation of state")

$$\rho = mn, \mathbf{u}, \sigma_{ij}^2$$

*Well-posedness not guaranteed, inversion to DF (almost) always non-trivial.*

# Star clusters as quasi-isothermal spheres

- Distribution function

$$f_K(E) = \begin{cases} A\left[\exp\left(-aE\right) - \exp\left(-aE_0\right)\right] & \text{if } E \leq E_0 \\ 0 & \text{if } E > E_0 \end{cases}$$

King AJ 1966

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_c$$

- Spherical symmetry: $\psi(r) = a[E_0 - \Phi_c(r)]$, with boundary set by $\psi(r_t) = 0$
- Physical quantities as velocity moments, e.g. density

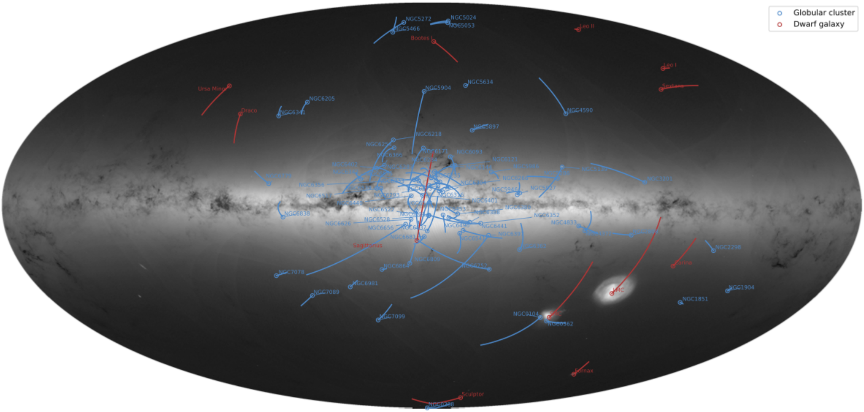
$$\rho(r) = \int f_K(E) d^3\boldsymbol{v} = \hat{A} e^{\psi} \gamma\left(\frac{5}{2}, \psi(r)\right) = \hat{A}\hat{\rho}(\psi(r))$$

- Self-consistency of mean-field potential via Poisson eq. $\nabla^2\Phi_c(r) = 4\pi G\rho(r)$
- Initial value problem; one parameter $(\Psi)$, two physical scales $(A, a)$

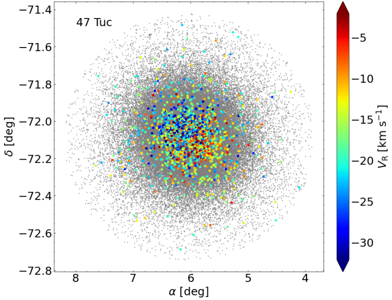
in dimensionless form, with $\hat{r} = r/r_0$ and $r_0 = [9/(4\pi G\rho_0 a)]^{1/2}$

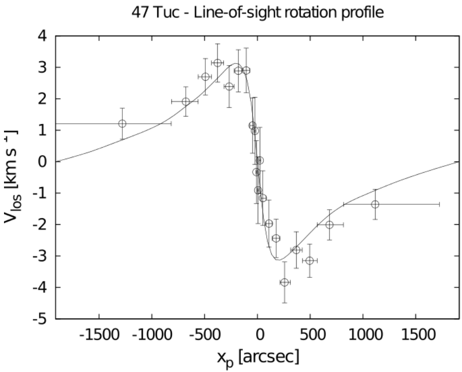
$$\nabla^2\psi = -9\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)}$$
$$\psi(0) = \Psi$$
$$\psi'(0) = 0$$

# A NEW ASTRONOMICAL LANDSCAPE: Gaia reveals our own Milky Way



# THE NEW PHASE SPACE COMPLEXITY OF OLD STELLAR SYSTEMS




Star clusters no longer
*just* simple "balls of stars":
they rotate, have velocity anisotropy ...

**Katz et al. A&A 2022 (Gaia DR3)**

When a GC rotation curve
was a heresy!

**Bianchini, Varri et al. A&A 2012**

# Rigidly rotating equilibria

# RIGIDLY ROTATING EQUILIBRIA: Definitions

Varri & Bertin 2012 A&A

- If total angular momentum is non-vanishing, in the derivation of Maxwell-Boltzmann distribution function: $E \rightarrow H = E - \omega J_z$, where $\omega$ represents the (rigid) angular velocity of the system.

Landau & Lifchitz Stat Phys 1967

- Distribution function:

$$f_K^r(H) = \begin{cases} A\left[\exp(-aH) - \exp(-aH_0)\right] & \text{if } H \leq H_0 \\ 0 & \text{if } H > H_0 \end{cases}$$

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \Phi_{cen} + \Phi_C \qquad \Phi_{cen}(\mathbf{r}) = -\frac{1}{2}\omega^2(x^2 + y^2)$$

$$\psi(\mathbf{r}) = a\{H_0 - [\Phi_c(\mathbf{r}) + \Phi_{cen}(\mathbf{r})]\}$$

Concentration $\leftrightarrow$ $\Psi \equiv \psi(\mathbf{0})$ Rotation strength $\leftrightarrow$ $\chi \equiv \dfrac{\omega^2}{4\pi G\rho_0}$

- Two domains separated by the boundary surface of the configuration, defined by $\psi(r) = 0$, which is unknown *a priori*.

$$\hat{\nabla}^2\psi = -9\left[\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)} - 2\chi\right] \quad \text{for} \quad \psi > 0 \quad \text{(Poisson)}$$

$$\hat{\nabla}^2\psi = 18\chi \quad \text{for} \quad \psi < 0 \quad \text{(Laplace)}$$

Elliptic PDE in a free boundary problem

# RIGIDLY ROTATING EQUILIBRIA: Perturbation approach

**Varri & Bertin 2008 ApJ**

- Rotation effect = (small) perturbation acting on the configuration described by the spherical (King 1966) models: $\chi \ll 1$

$$\psi(\hat{\mathbf{r}};\chi) = \sum_{k=0}^{\infty} \frac{1}{k!} \psi_k(\hat{\mathbf{r}})\, \chi^k$$

- Expansion of the general term of the series $\psi_k(\hat{\mathbf{r}})$ in Legendre polynomials $\rightarrow$ one-dimensional (radial) initial-value problems.

- This perturbation problem is singular!
  The convergence radius of the asymptotic series vanishes $\hat{r} \rightarrow \hat{r}_{tr}$, i.e. the validity of the expansion breaks down when $\psi_0 = \mathcal{O}(\chi)$.
  - Introduction of an intermediate region (boundary layer)
  - Asymptotic matching à la Van Dyke for $(\psi^{(\mathrm{int})}, \psi^{(\mathrm{lay})})$ and $(\psi^{(\mathrm{lay})}, \psi^{(\mathrm{ext})})$

**Van Dyke, Perturbation Methods in Fluid Mechanics, 1975**

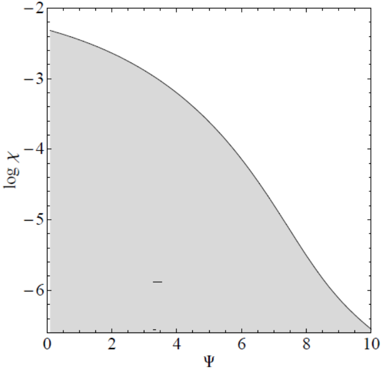
- Inspiration: rigidly rotating polytropes — **Chandrasekhar MNRAS 1933, ... Smith Ap&SS 1975**

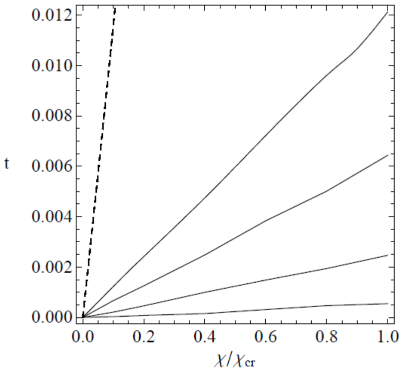
- Full explicit solution to 2 orders in $\chi$.

- *By induction*, the $k$-th order solution $\psi^{(k)}$ contains only the $l = 0, 2, .., 2k$ polynomials.

# RIGIDLY ROTATING EQUILIBRIA: Parameter space

Varri & Bertin A&A 2012




Ask me about stability!

$t = T/|W|$

(i.e., total ordered kinetic energy)

$t(e) \sim \chi(e) \sim \frac{2}{15}e^2$ for $e << 1$

Maclaurin sequence of ellipsoids (dashed)

# Rigidly rotating equilibria: Properties

Varri & Bertin A&A 2012

Deformation shaped by the centrifugal potential: "elongation" on $(\hat{x}, \hat{y})$

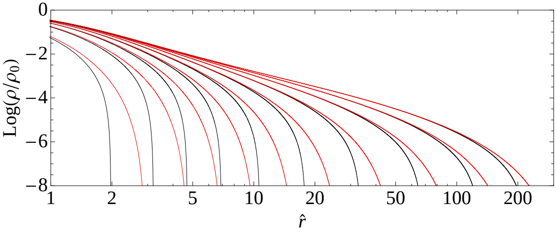
$$e = [1 - (\hat{b}/\hat{a})^2]^{1/2}$$
$$\hat{a} \geq \hat{b}$$

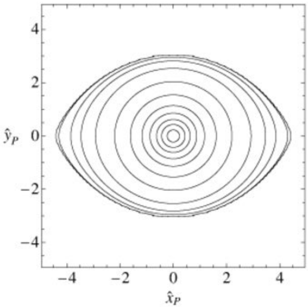
$$e_0 = \mathcal{O}(\chi^{1/2})$$
Nontrivial!

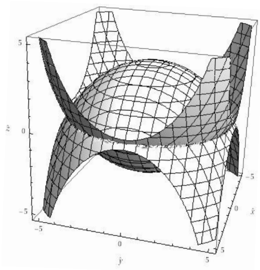
Quadrupole moments calculated analytically!

$$\hat{Q}_{zz}/\hat{Q}_{xx} = -2$$
$$\hat{Q}_{yy}/\hat{Q}_{xx} = 1$$
for every $\chi$ and $\Psi$

$\Psi = 1, \ldots, 10 \qquad \chi = \chi_{cr}(\Psi)$



# Equilibria with central black holes

# EQUILIBRIA WITH BH: Boundary conditions

Bonsor, Varri & Vanneste, almost sub.

- Again, start from spherical (King 1966) equilibria
- Introduce black hole via "hydrostatic equilibrium" condition **Huntley & Saslow ApJ 1975**

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM_{BH}}{r^2} - \frac{4\pi G}{r^2}\int_{r_{min}}^{r} s^2\rho(s)ds$$

- New initial value problem

$$\nabla^2\psi = -9\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)}$$
$$\psi(\epsilon) = \Psi$$
$$\psi'(\epsilon) = -\frac{9\mu}{4\pi\epsilon^2}$$

- Three dimensionless parameters (and two physical scales)
  - $\Psi$ concentration, or rather depth of the central potential well
  - $\mu$ mass of the central black hole
  - $\epsilon$ inner radius

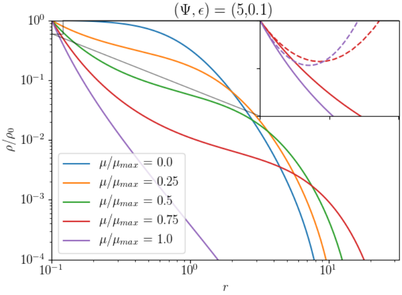
# EQUILIBRIA WITH BH: Main properties

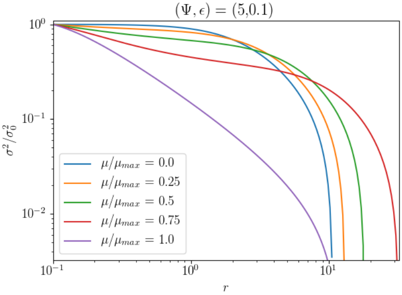
Bonsor, Varri & Vanneste, almost sub.

- Numerical solution of initial value problem is trivial
- Substantial central slopes in both density and velocity dispersion
- For given $\Psi$ and $\epsilon$, a maximum value $\mu_{max}$ exists, set by

$$a_0 = \Psi - \frac{9\mu}{4\pi\epsilon} > 0$$

- Beyond a critical value $\mu > \mu_c$, the equilibria change their structure!



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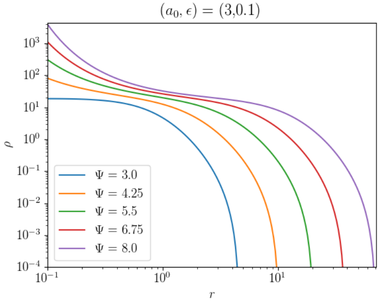
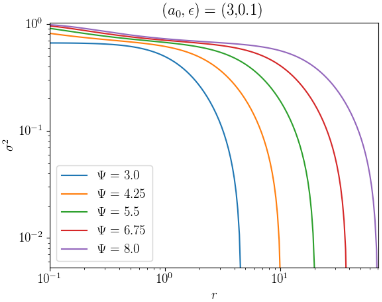
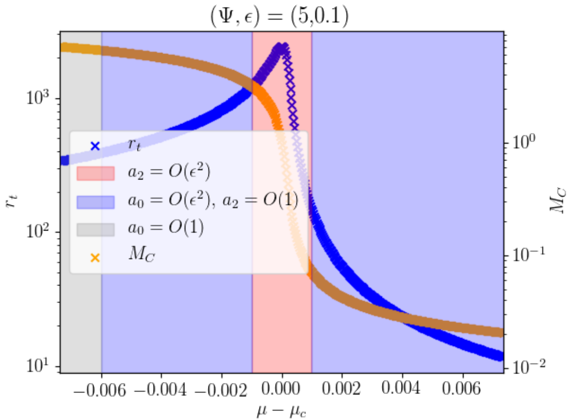


# Equilibria with BH: Three nested asymptotic regimes




- $r_t$ boundary radius; $M_C = -\mu - \dfrac{4\pi}{9} r_t^2 \left.\dfrac{\mathrm{d}\psi}{\mathrm{d}r}\right|_{r=r_t}$ mass of the star cluster
- $a_n$ constant of integration at order $n$ of the expansion
- Different *regions* in each *regime*
  - Region I: rescale $r_1 = \hat{r}/\epsilon$ and expand $\psi$ as a series to investigate the central behaviour
  - Subsequent regions when the previous one breaks down, pushing the solution further away from the centre

# EQUILIBRIA WITH BH: Regime I: $a_0 = \mathcal{O}(1)$

- Rescale radial variable, expand in series of $\epsilon$, and solve initial value problems at each order

$$\psi^{(I)}(r_1) = \psi_0^{(I)}(r_1) + \epsilon\psi_1^{(I)}(r_1) + \epsilon^2\psi_2^{(I)}(r_1) + \cdots .$$

- Region I

$$\psi^{(I)}(r_1) = \underbrace{\left(\Psi - \frac{9\mu}{4\pi\epsilon}\right)}_{a_0} + \frac{9\mu}{4\pi\epsilon}\frac{1}{r_1} + \epsilon^2\psi_2^{(I)}(r_1)$$

where

$$\nabla^2_{r_1}\psi_2^{(I)}(r_1) = -9\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)}$$
$$\psi_2^{(I)}(1) = 0$$
$$\psi_2^{(I)\prime}(1) = 0$$

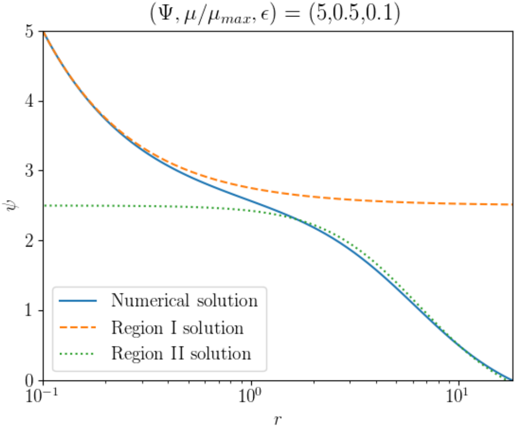
- Leading order solution dominated by the black hole, the stellar component acts as a 'tracer' at the next order
- Large radius approximation of $\psi_2^{(I)}$ can be obtained, when the solution becomes invalid

# Equilibria with BH: Regime I: $a_0 = \mathcal{O}(1)$

- Region II

$$\nabla_r^2 \psi^{(II)} = -9\frac{\hat{\rho}(\psi)}{\hat{\rho}(\Psi)}$$
$$\psi^{(II)}(0) = \lim_{r_1\to\infty} \psi^{(I)} = a_0$$
$$\psi^{(II)\prime}(0) = \lim_{r_1\to\infty} \psi^{(I)\prime} = 0$$

- Boundary condition set via asymptotic matching with the Region I solution
- Recognisable as a classical King model with a reduced central concentration of $a_0$



## EQUILIBRIA WITH BH: Regime II: $a_0 = \mathcal{O}(\epsilon^2)$, $a_2 = \mathcal{O}(1)$

- Region I: significant change in the form of the leading order

$$\psi^{(I)} = \frac{\Psi}{r_1} + \epsilon^2 \psi_2^{(I)}$$

such that

$$\nabla^2_{r_1} \psi_2^{(I)}(r_1) = -9 \frac{\hat{\rho}(\Psi/r_1)}{\hat{\rho}(\Psi)}$$
$$\psi_2^{(I)}(1) = 0$$
$$\psi_2^{(I)\prime}(1) = \epsilon^{-2} a_0$$

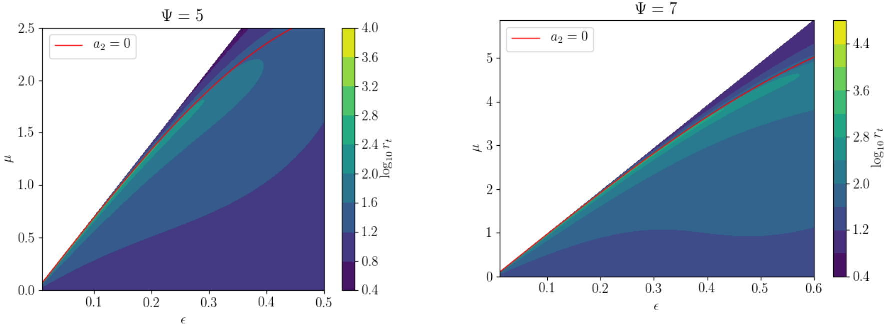
- Constant of integration associated with the above problem

$$a_2 = \epsilon^{-2} a_0 - \frac{9}{\hat{\rho}(\Psi)} \int_1^\infty s \hat{\rho}\left(\frac{\Psi}{s}\right) \mathrm{d}s$$

which may be positive or negative

as opposed to $a_0$, which had to be strictly positive

# EQUILIBRIA WITH BH: Regime II: $a_0 = \mathcal{O}(\epsilon^2)$, $a_2 = \mathcal{O}(1)$




- Condition $a_2 = 0$ provides a good criterion for the transition between the two types of equilibria
- Regime II asymptotics offers a good description on both sides of the transition

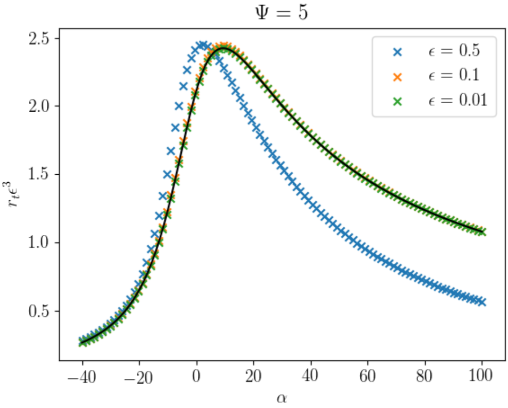
# EQUILIBRIA WITH BH: Regime III: $a_0 = \mathcal{O}(\epsilon^2)$, $a_2 = \mathcal{O}(\epsilon^2)$

- Region I: same approach as before, but 4th order solution is required to match properly
- Region II: requires further scaling $r_2 = \epsilon^4 r_1$ and $\psi^{(II)} = \epsilon^4 \phi$
- Initial value problem

$$\nabla^2_{r_2} \psi^{(II)} = -\kappa \phi^{5/2}$$

where $\kappa = \dfrac{5}{18\rho(\hat{\Psi})}$
from $\gamma\,(5/2, \psi) \approx \frac{2}{5}\psi^{5/2}$

- Boundary condition $\phi \to \alpha + F(\Psi, r_2)$ as $r_2 \to 0$ with $\alpha = a_4 + 40\kappa^2\Psi^4 \ln(\epsilon)$

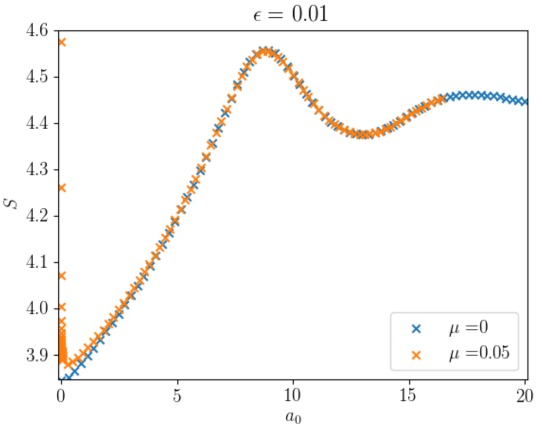



$$\mu_c = \frac{4\pi\epsilon}{9}\left(\Psi + \epsilon^2 C(\Psi) + \epsilon^4 \ln(\epsilon) 40\kappa^2\Psi^4 - \epsilon^4\left[\alpha_c - D(\Psi)\right]\right)$$

# EQUILIBRIA WITH BH: Entropy

$$S = -\int f(\boldsymbol{x}, \boldsymbol{v}) \ln(f(\boldsymbol{x}, \boldsymbol{v})) \mathrm{d}^3\boldsymbol{v}\mathrm{d}^3\boldsymbol{x}.$$

- Regime I: the equilibria behave like King 1966 models with central concentration $a_0$
- For $a_0 = \mathcal{O}(1)$, the classic oscillatory behaviour is recovered
- Rapid increase in entropy for small values of $a_0$



# Equilibria with BH: Entropy

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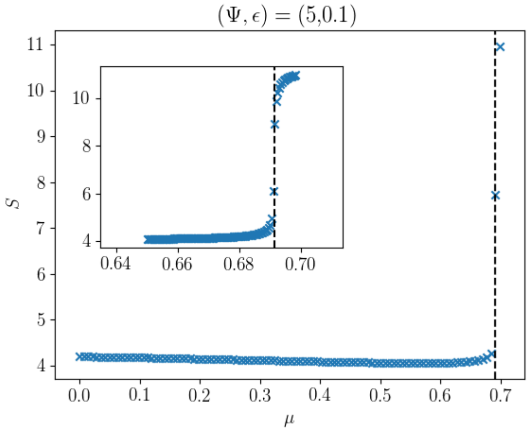
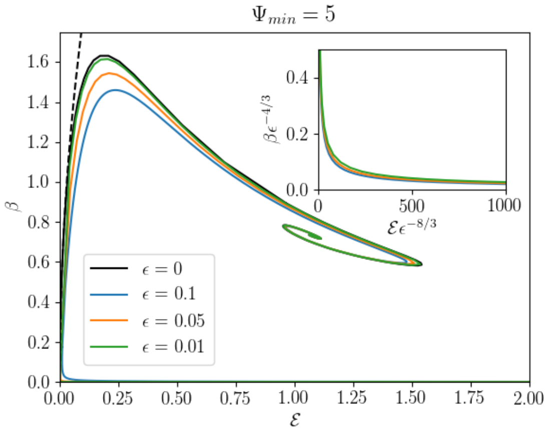


# Equilibria with BH: "Caloric curve"

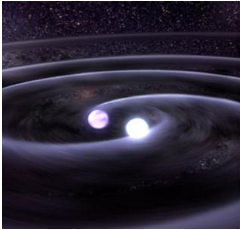
$$\beta \equiv aG^2 M_C^{4/3} (8\sqrt{2}\pi A e^{-aE_0})^{2/3} = \frac{81 \cdot 3^{2/3}}{16\pi^2} \frac{\hat{M}_C^{4/3}}{\hat{\rho}(\Psi)^{2/3}}$$

$$\mathcal{E} \equiv -\frac{E_{tot}}{G^2 M_C^{7/3} (8\sqrt{2}\pi A e^{-aE_0})^{2/3}} = -\frac{2\pi}{3^{8/3}} \frac{\hat{\rho}(\Psi)^{2/3} \hat{U}}{\hat{M}_C^{7/3}}$$

- Classic "cold spiral" for large $\Psi$, which denotes gravothermal catastrophe
- New branch for small $a_0$!
- Regime III asymptotics offers the correct scaling around the new phase transition! (1st order microcanonical)



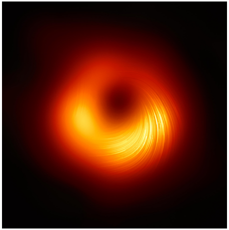
# Why all this?



Small, “stellar-mass”
black holes exist!

Gravitational waves
February 2016

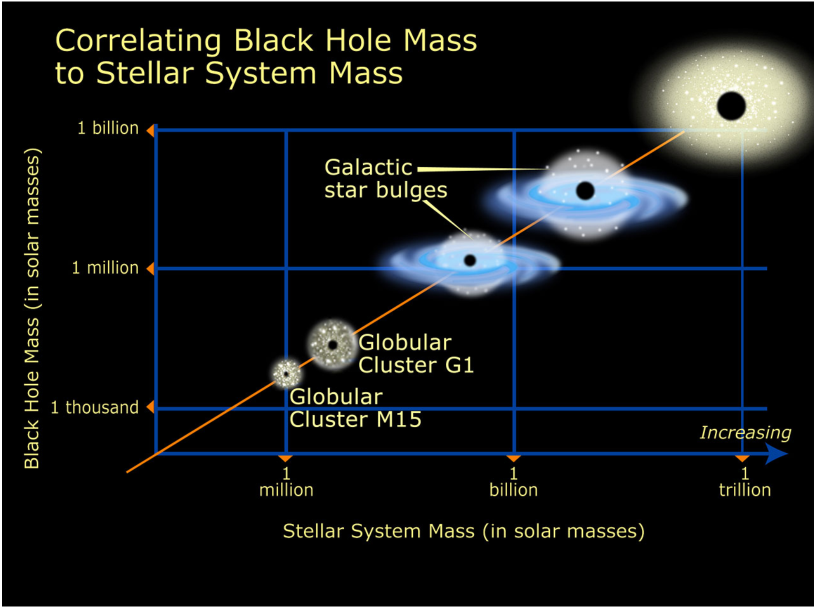
“Intermediate-mass”
black holes??



Large, “supermassive”
black holes exist!

Event Horizon Telescope
April 2019, May 2022

# Yet, no detection!



# Parting thoughts

- The dynamics of (small) stellar systems has much to say about some of the biggest questions in contemporary astrophysics. Ask me more.

- "Phase space complexity" essential to exploit new-generation astronomical data. Rigidly rotating equilibria as a minimal example. Ask me more.

- Quasi-isothermal spheres with central black holes proposed.
  New phase transition identified, promising observables.
  Self-consistent treatment of the mean-field potential is key!

- Kinetic theory of self-gravitating systems (and plasmas) still is a goldmine.
  *KITP program in Summer 2024*