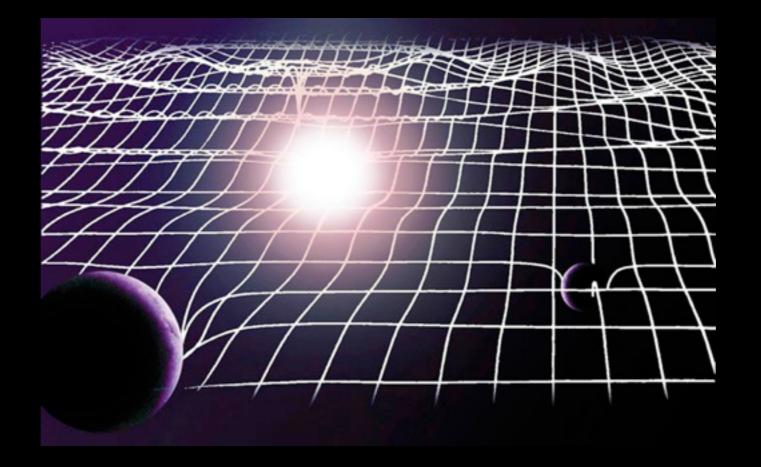
Gravitational Wave Astrophysics

Johan Samsing Princeton University



Part I



Gravitational Wave Properties



Frequency:

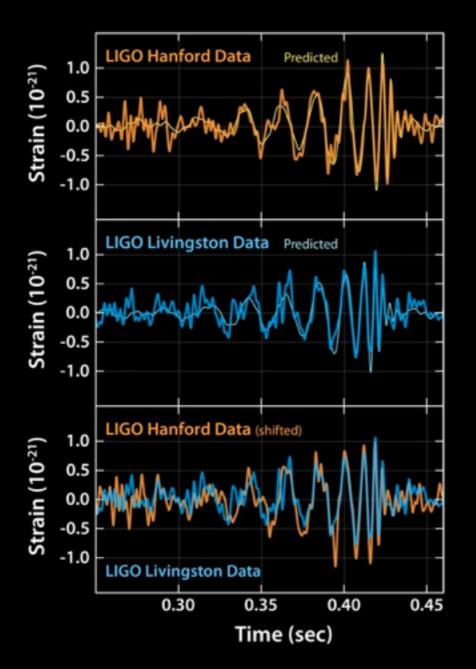


Strain:

$$h = 5 \times 10^{-22} \left(\frac{M}{2.8 M_{\odot}}\right)^{2/3} \left(\frac{\mu}{0.7 M_{\odot}}\right) \left(\frac{f}{100 \text{Hz}}\right)^{2/3} \left(\frac{15 \text{Mpc}}{r}\right)^{2/3} \left(\frac{1$$

Merger time:

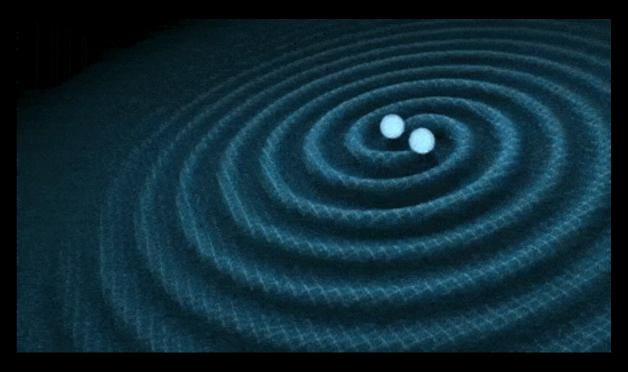
 $t_{life} \propto a^4 m^{-3} (1-e^2)^{7/2}$



<u> Objects:</u>	
NS Binary:	$f_{GW}(NS)$
VD Binary:	$f_{GW}(WL)$
Solar Binary:	f _{GW} (su

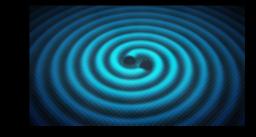
 $f_{GW}(NS) \sim 10^3 \text{ Hz}$ $f_{GW}(WD) \sim 10^{-1} \text{ Hz}$ $f_{GW}(sun) \sim 10^{-4} \text{ Hz}$

In an expanding U: redshift, (1+z)





Gravitational Wave Properties



Frequency:

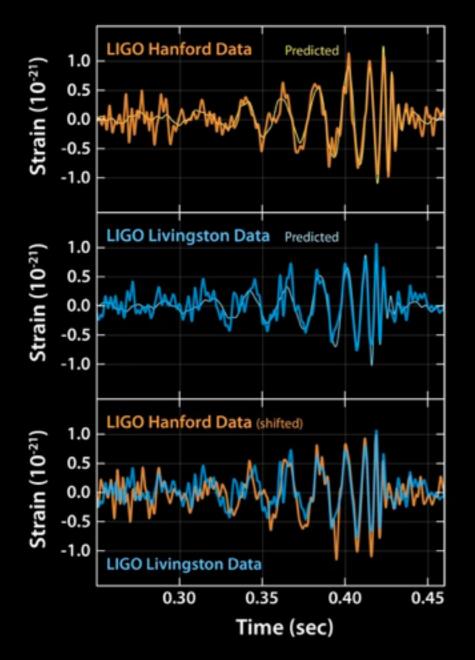


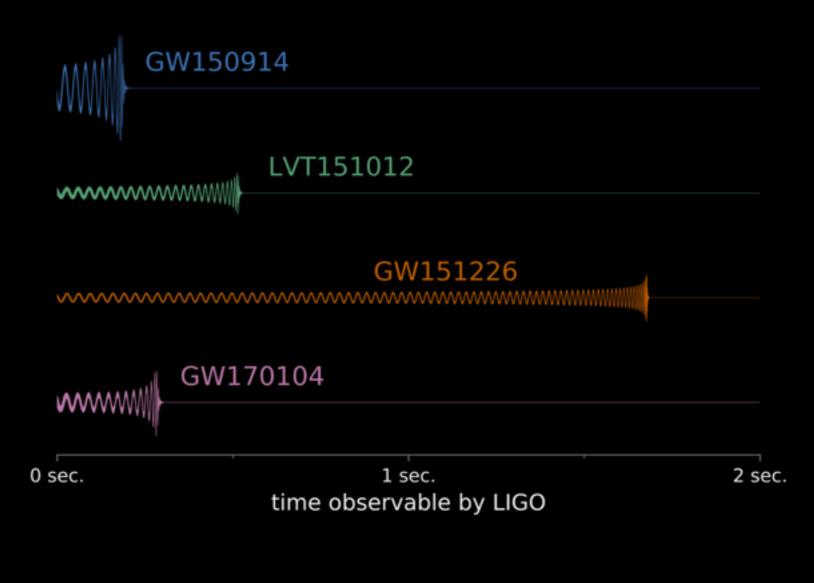
Strain:

$$h = 5 \times 10^{-22} \left(\frac{M}{2.8 M_{\odot}}\right)^{2/3} \left(\frac{\mu}{0.7 M_{\odot}}\right) \left(\frac{f}{100 {\rm Hz}}\right)^{2/3} \left(\frac{15 {\rm Mpc}}{r}\right)^{1/3} \left(\frac{15 {\rm$$

Merger time:

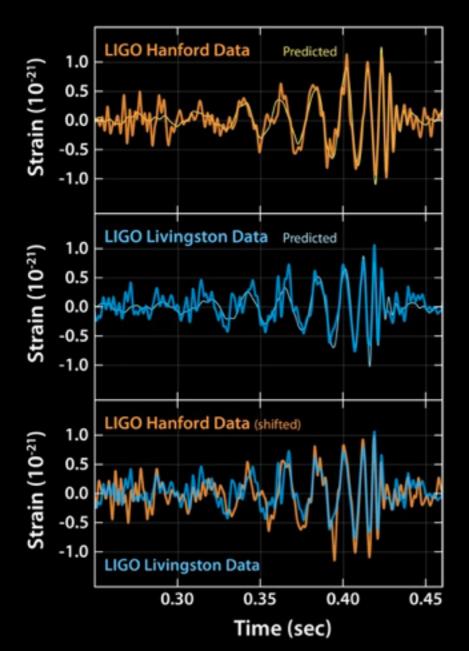
$$t_{life} \propto a^4 m^{-3} (1-e^2)^{7/2}$$

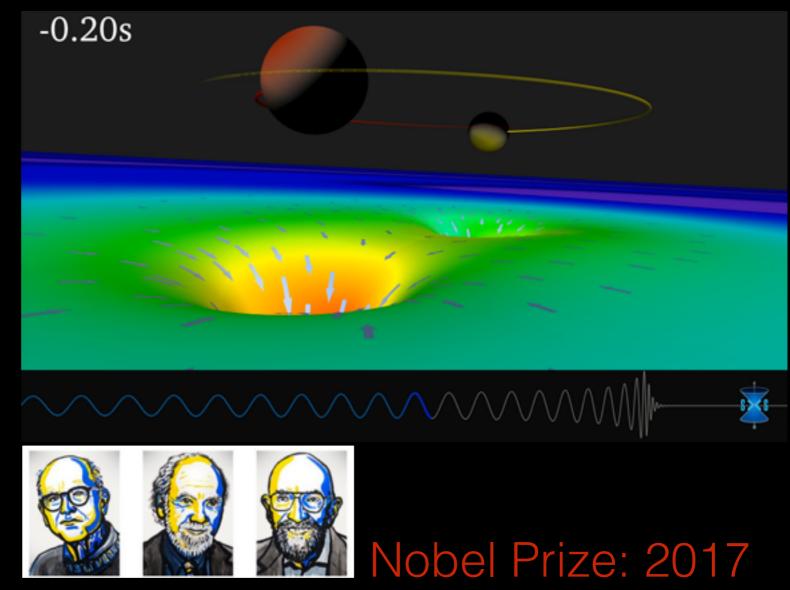




Gravitational Waves have now been observed!

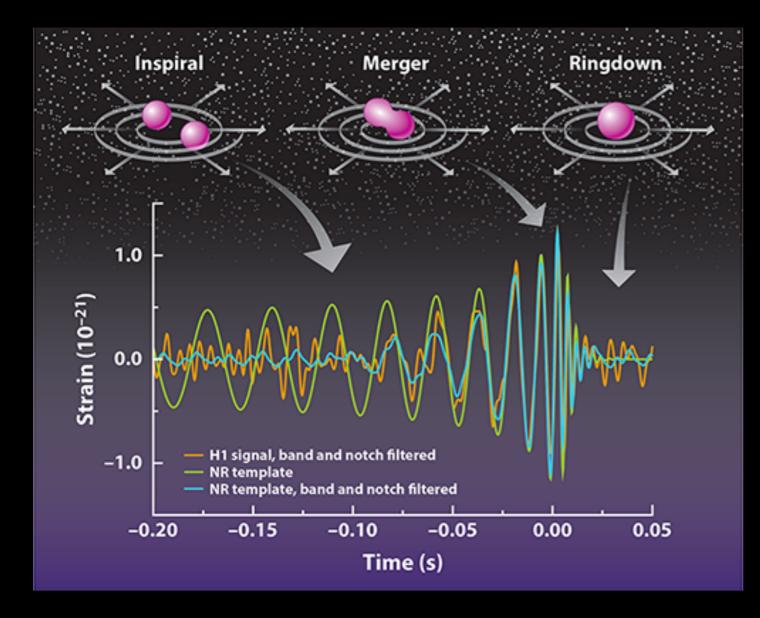
Real Signal: Animation:

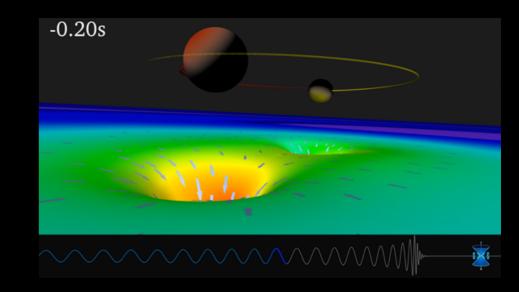




Dynamics, Merger, and Simulation

The three stages of merger:



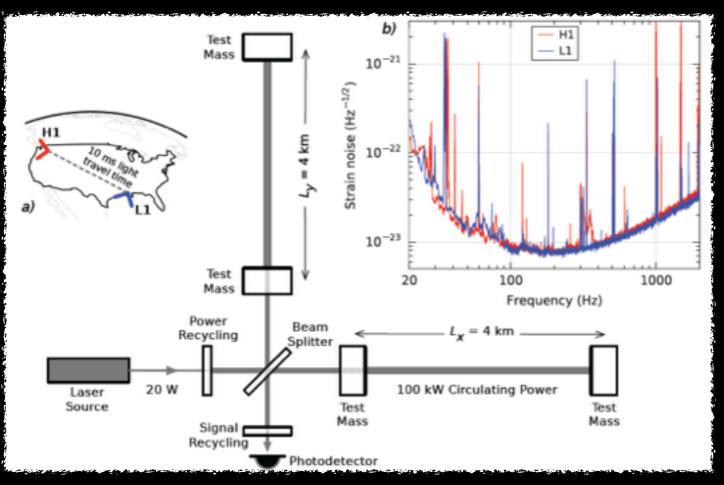


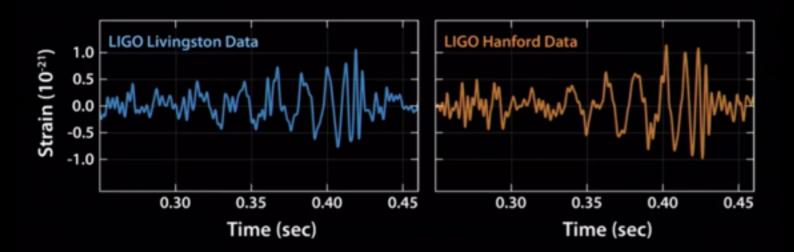


First GR sim.

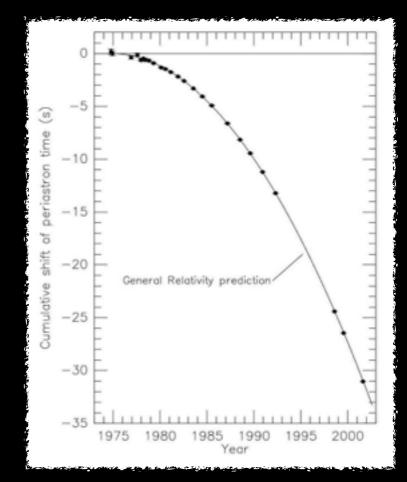
How Gravitational Waves are Detected

Direct:

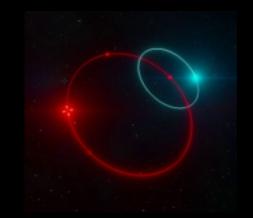


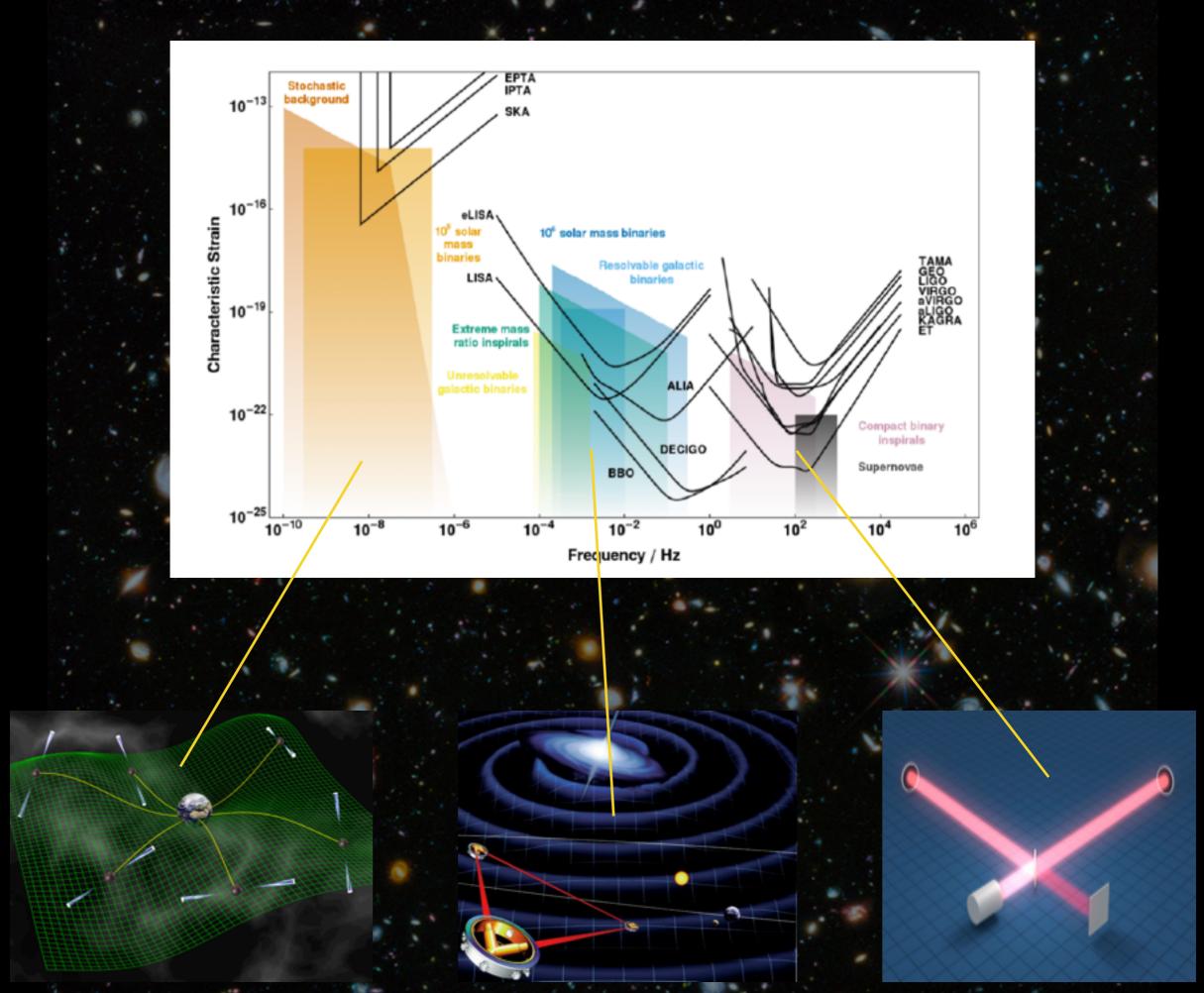


Indirect:



Hulse-Taylor Pulsar

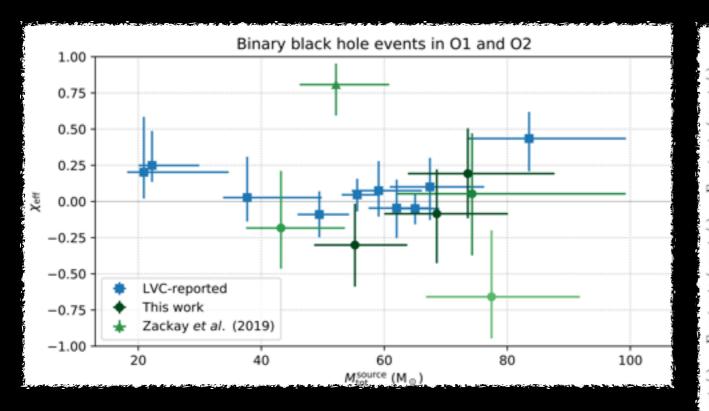




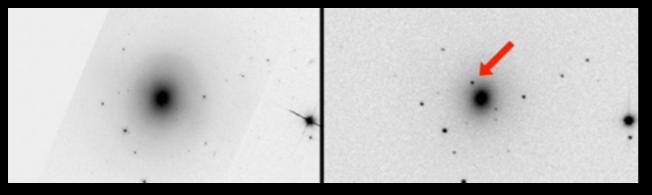
Current Status: Any Surprises yet?

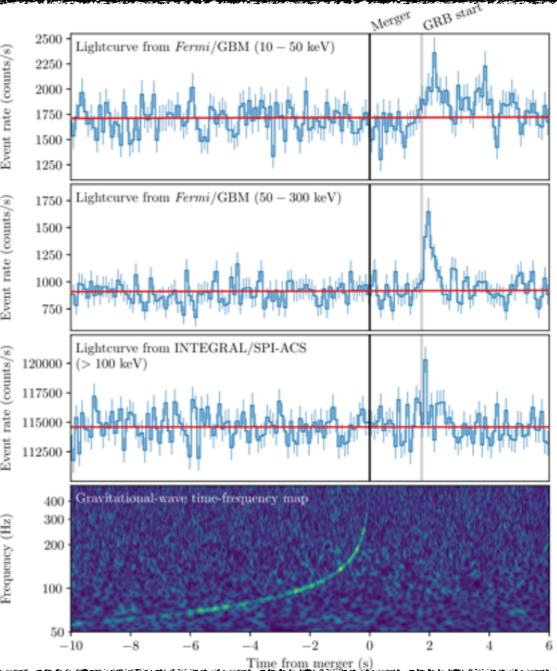
Black Holes:

Neutron Stars:



NS-NS optical signal:





What observations are we waiting for?

How do we know its Black Holes?

Solar Binary:

 $f_{GW}(sun) \sim 10^{-4} \text{ Hz}$

WD Binary:

 $f_{GW}(WD) \sim 10^{-1} \text{ Hz}$

NS/BH Binary: $f_{GW}(NS)$

 $f_{GW}(NS) \sim 10^3 \text{ Hz}$

- What we measure:
- Masses
- Spins
- Eccentricity
- Dist./Pos./Redshift
- Acceleration

- What we can learn about:
- Gravity
- Space time: Extra Dim?
- Cosmology: Expansion
- Nuclear Physics
- Astrophysics

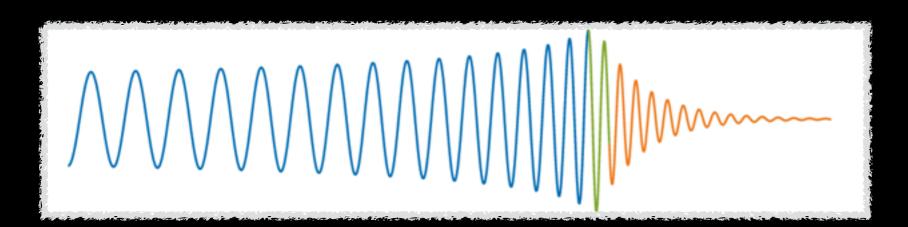
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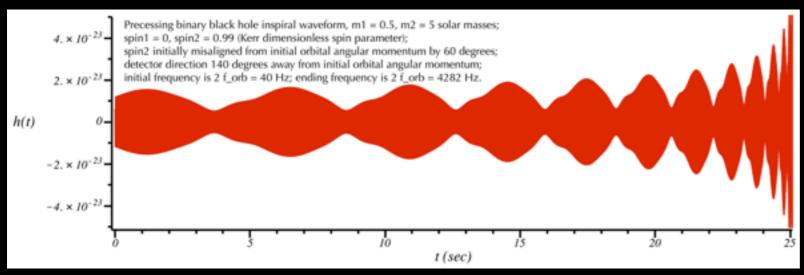
Generic Inspiral:

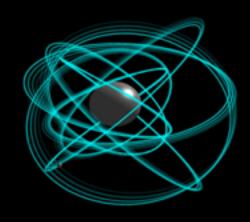


- What we measure:
- Masses
- Spins
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- Dist./Pos./Redshift
- Acceleration

- What we can learn about:
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- Astrophysics

Highly Spinning:





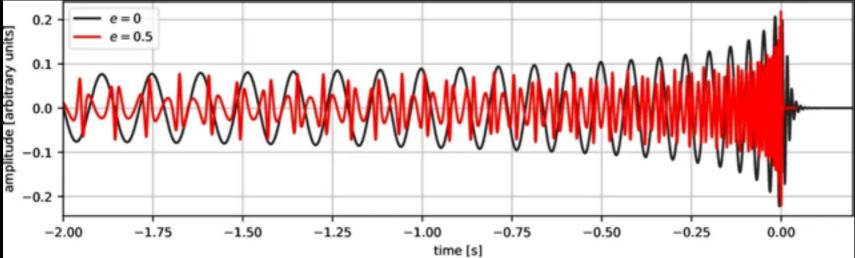
- What we measure:

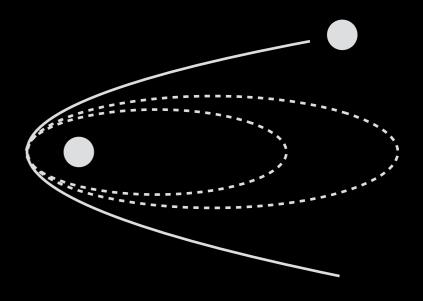
- Masses
- Spins
- Eccentricity
- Dist./Pos./Redshift
- Acceleration

- What we can learn about:

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- Space time: Extra Dim?
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- Nuclear Physics
- Astrophysics







Simulation:

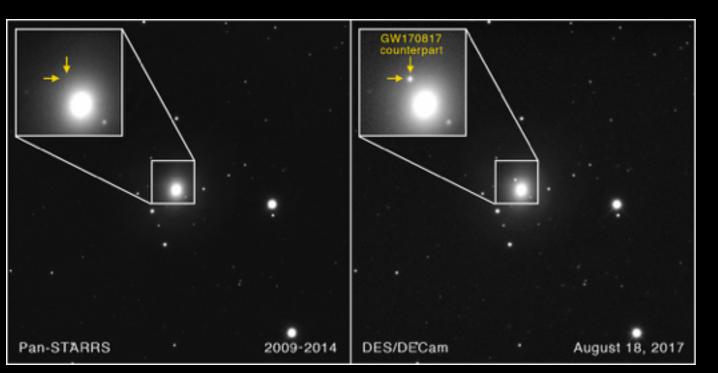
- What we measure:

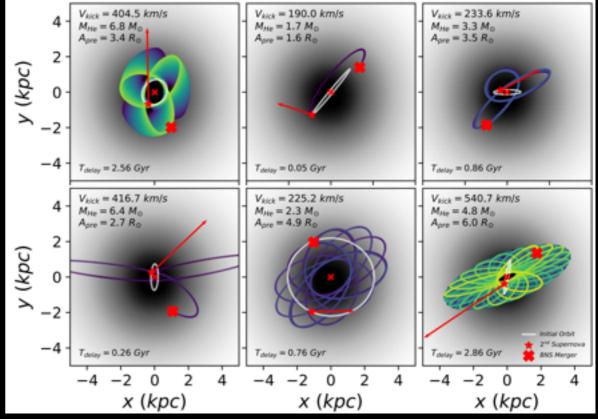
- Masses
- Spins
- Eccentricity
- Dist./Pos./Redshift
- Acceleration

- What we can learn about:

- Gravity
- Space time: Extra Dim?
- Cosmology: Expansion
- Nuclear Physics
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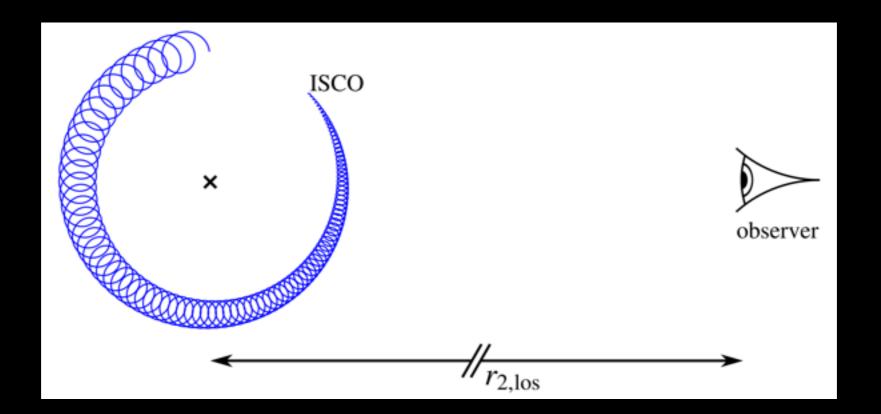
Observation:





- What we measure:
- Masses
- Spins
- Eccentricity
- Dist./Pos./Redshift
- Acceleration

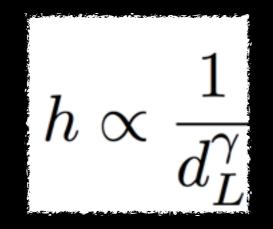
- What we can learn about:
- Gravity
- Space time: Extra Dim?
- Cosmology: Expansion
- Nuclear Physics
- Astrophysics



- What we measure:

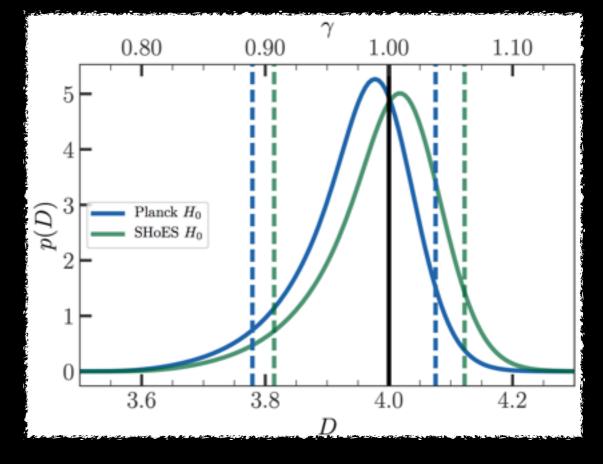
- Masses
- Spins
- Eccentricity
- Dist./Pos./Redshift
- Acceleration

Constraining the number of space-time dimensions:



- What we can learn about:

- Gravity (modify waveform)
- Space time: Extra Dim?
- Cosmology: Expansion
- Nuclear Physics
- Astrophysics



- What we measure:

- Masses
- Spins
- Eccentricity
- Dist./Pos./Redshift
- Acceleration

Hubble's law:

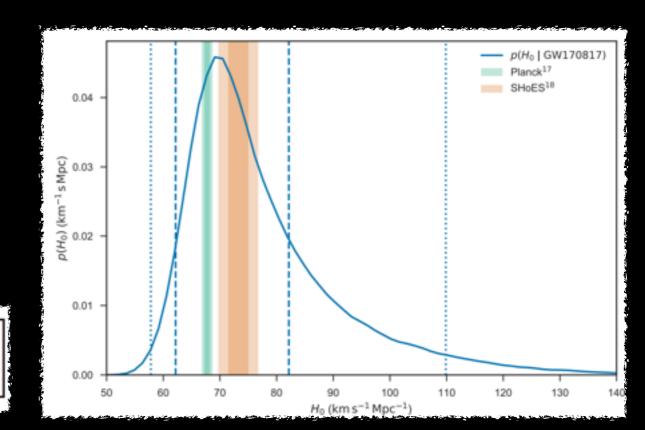
$$v_H = H_0 d$$

The cosmological params. can also be measured!

$$D_L(z) = \frac{c\left(1+z\right)}{H_0\sqrt{\Omega_K}} \sinh\left[\sqrt{\Omega_K} \int_0^z \frac{H_0}{H(z')} dz'\right]$$

- What we can learn about:

- Gravity
- Space time: Extra Dim?
- Cosmology: Expansion
- Nuclear Physics
 - Astrophysics



- What we measure:

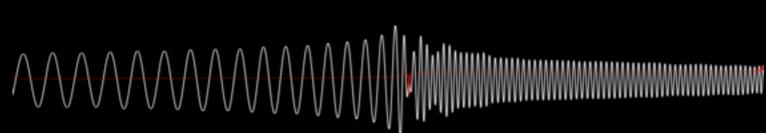
- Masses
- Spins
- Eccentricity
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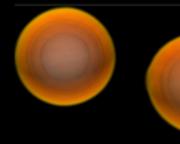
- What we can learn about:

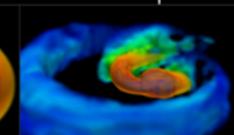
- Gravity
- Space time: Extra Dim?
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- Nuclear Physics
- Astrophysics

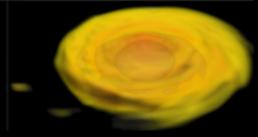
Tidal coupling and disruption:

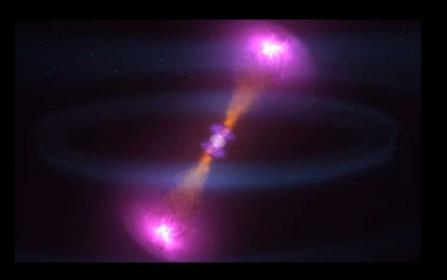
Jet/Enrichment:











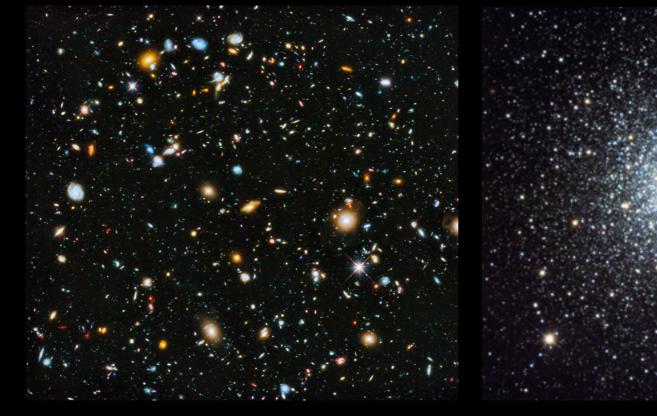
- What we measure:
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- Acceleration

- What we can learn about:
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Astrophysics: How do Black Holes Form and Merge?

- Primordial BHs
- Galactic Nuclei
- Supernovae
- Isolated BSE
- 3,4-body BSE
- AGN Disk
- NS Mountains
- ... ?



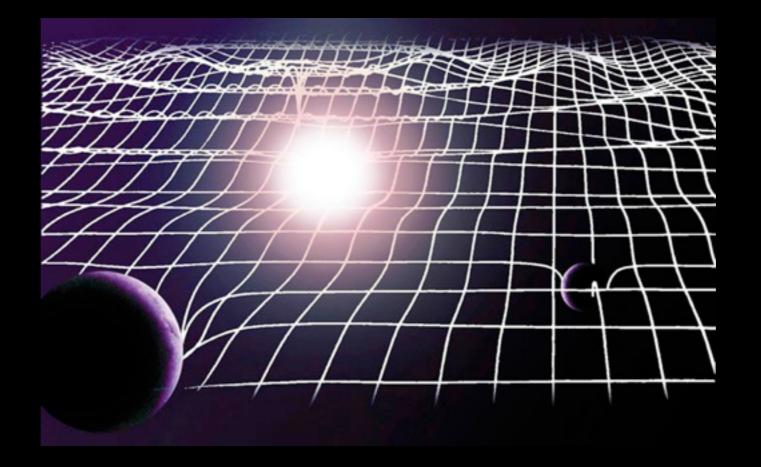






Gravitational Wave Astrophysics

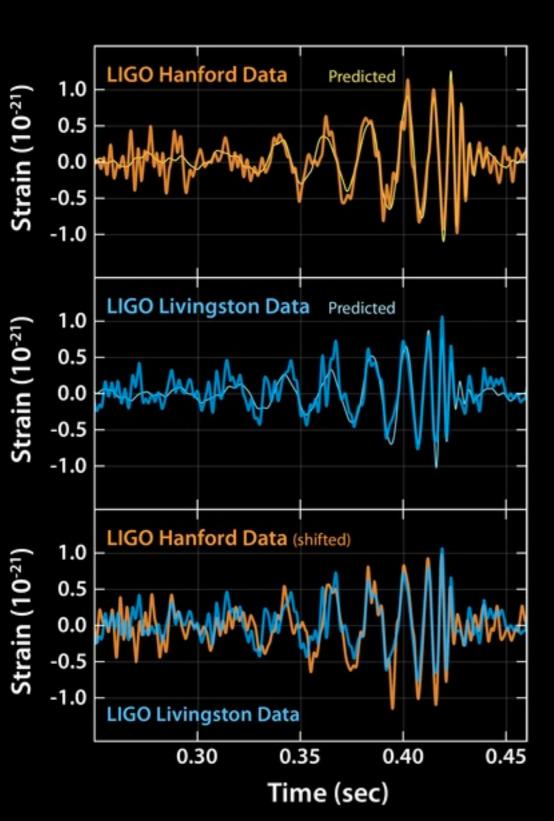
Johan Samsing Princeton University

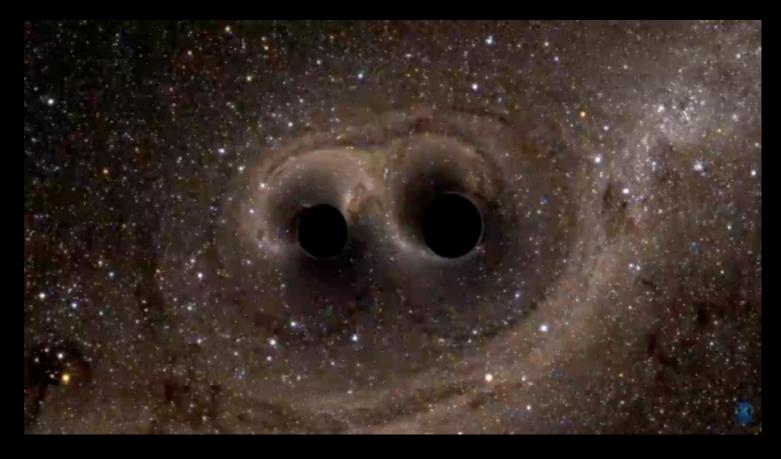




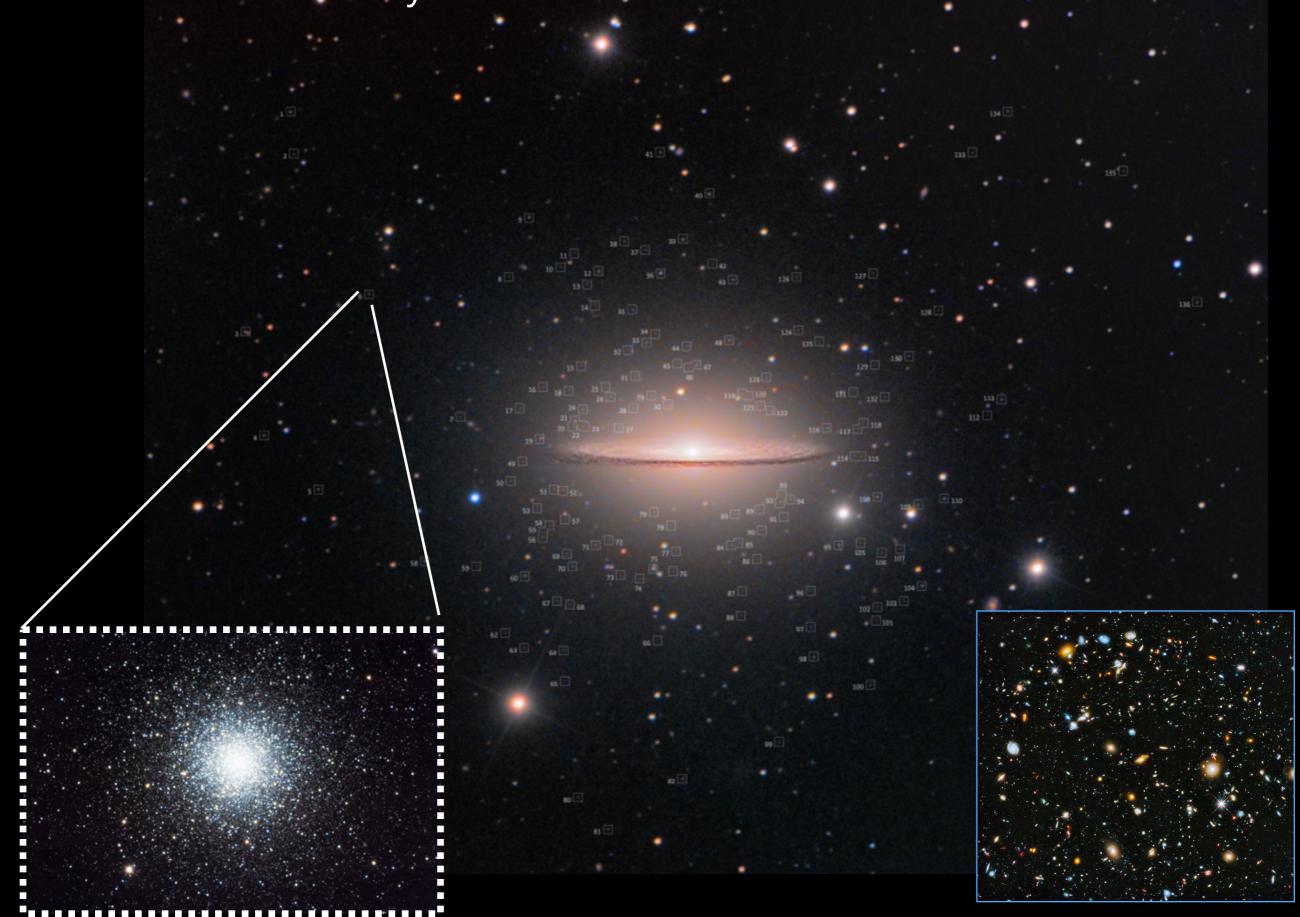
Observation:

Simulation:

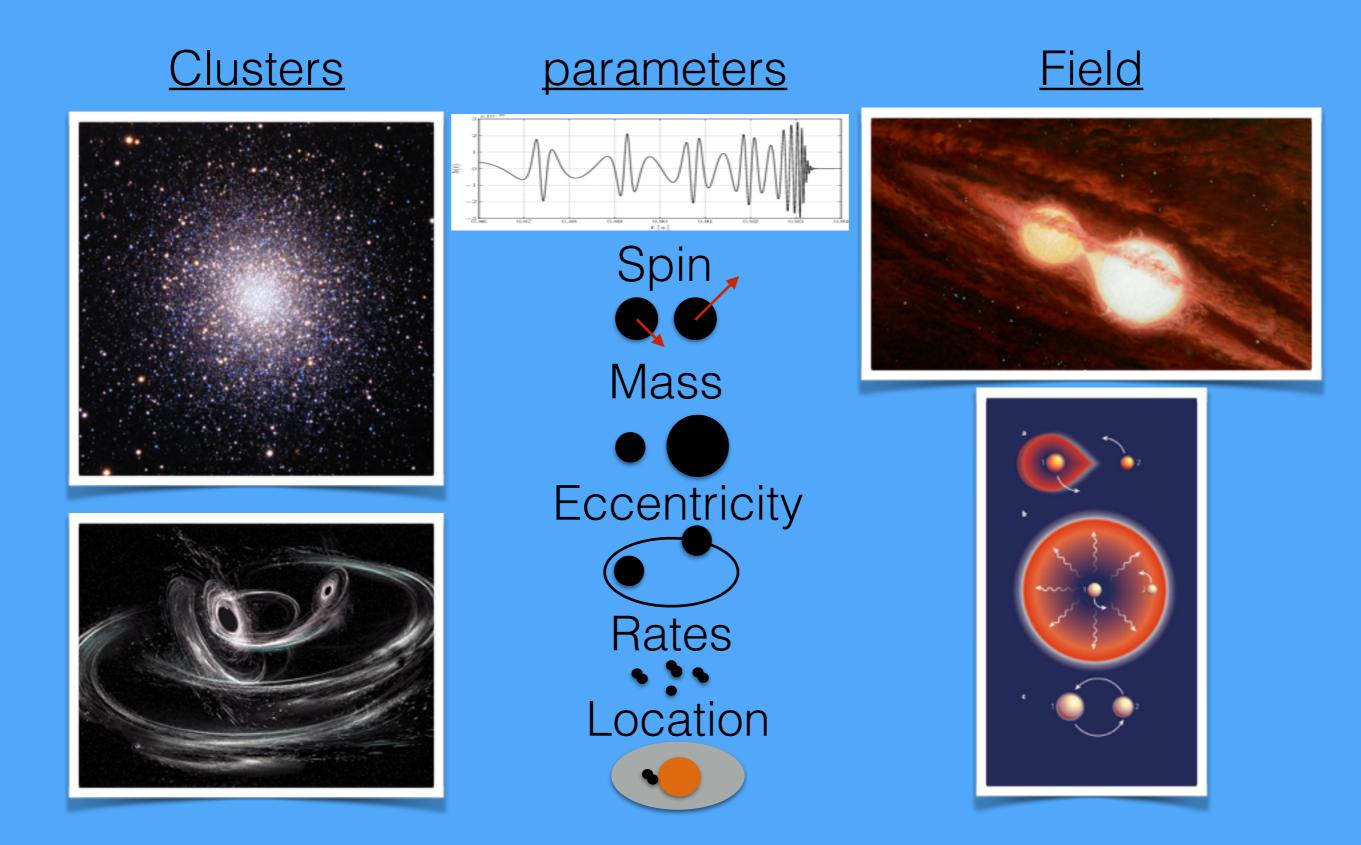




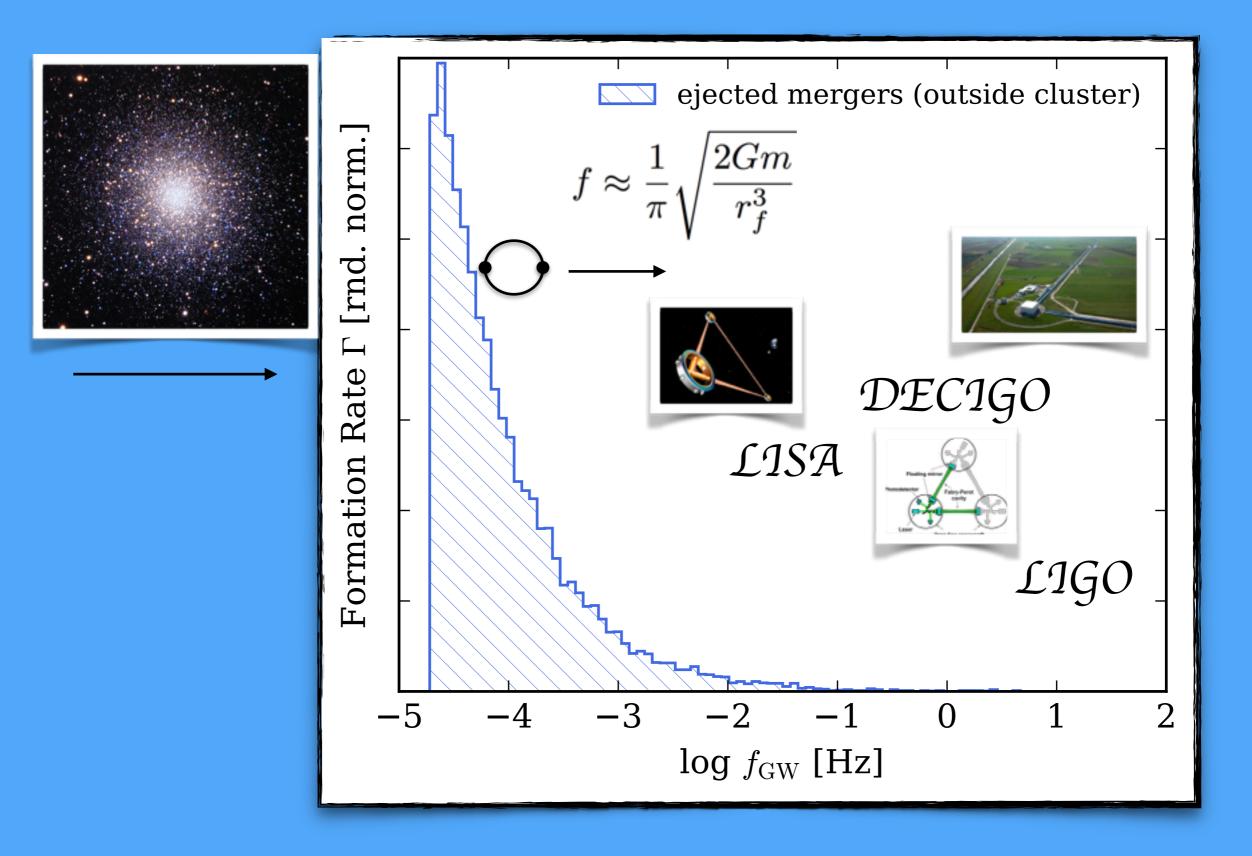
But where do they come from?



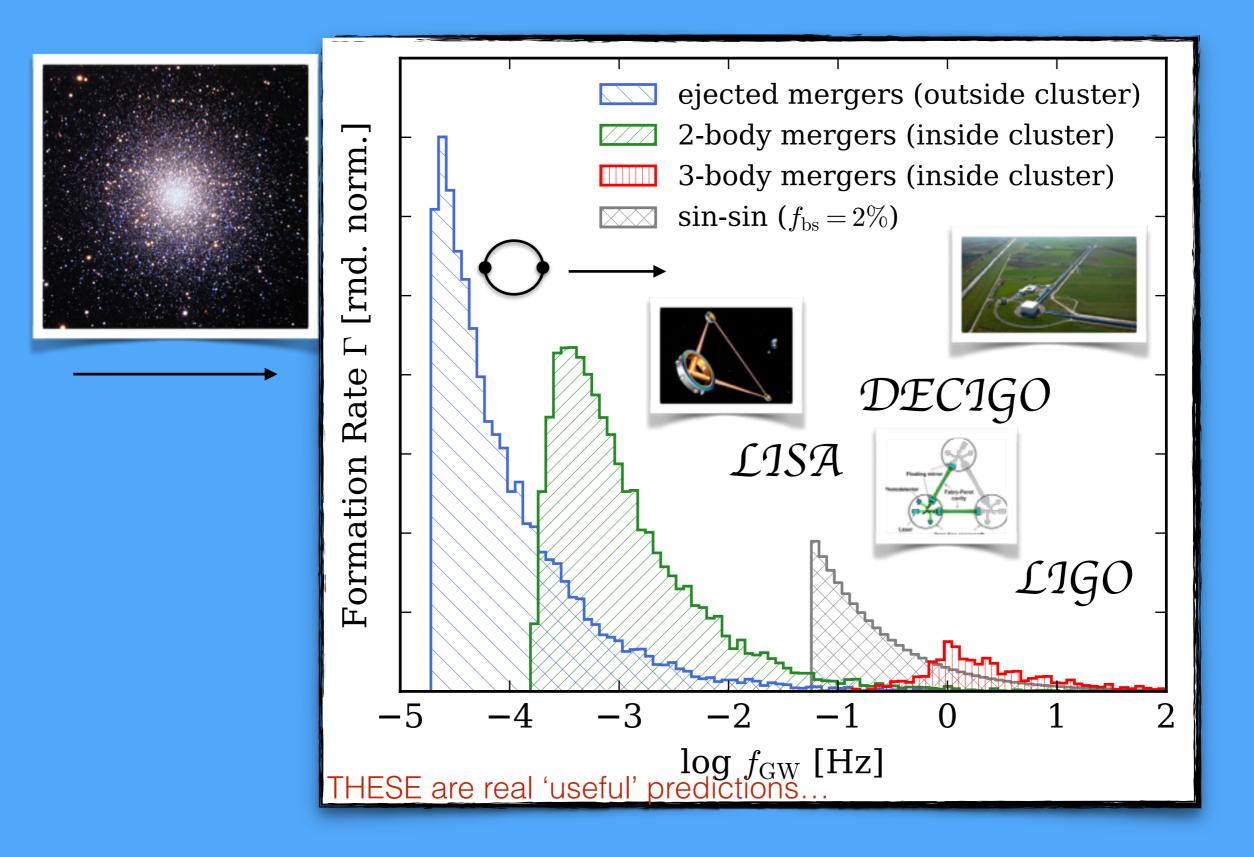
What is the origin of BBH mergers?

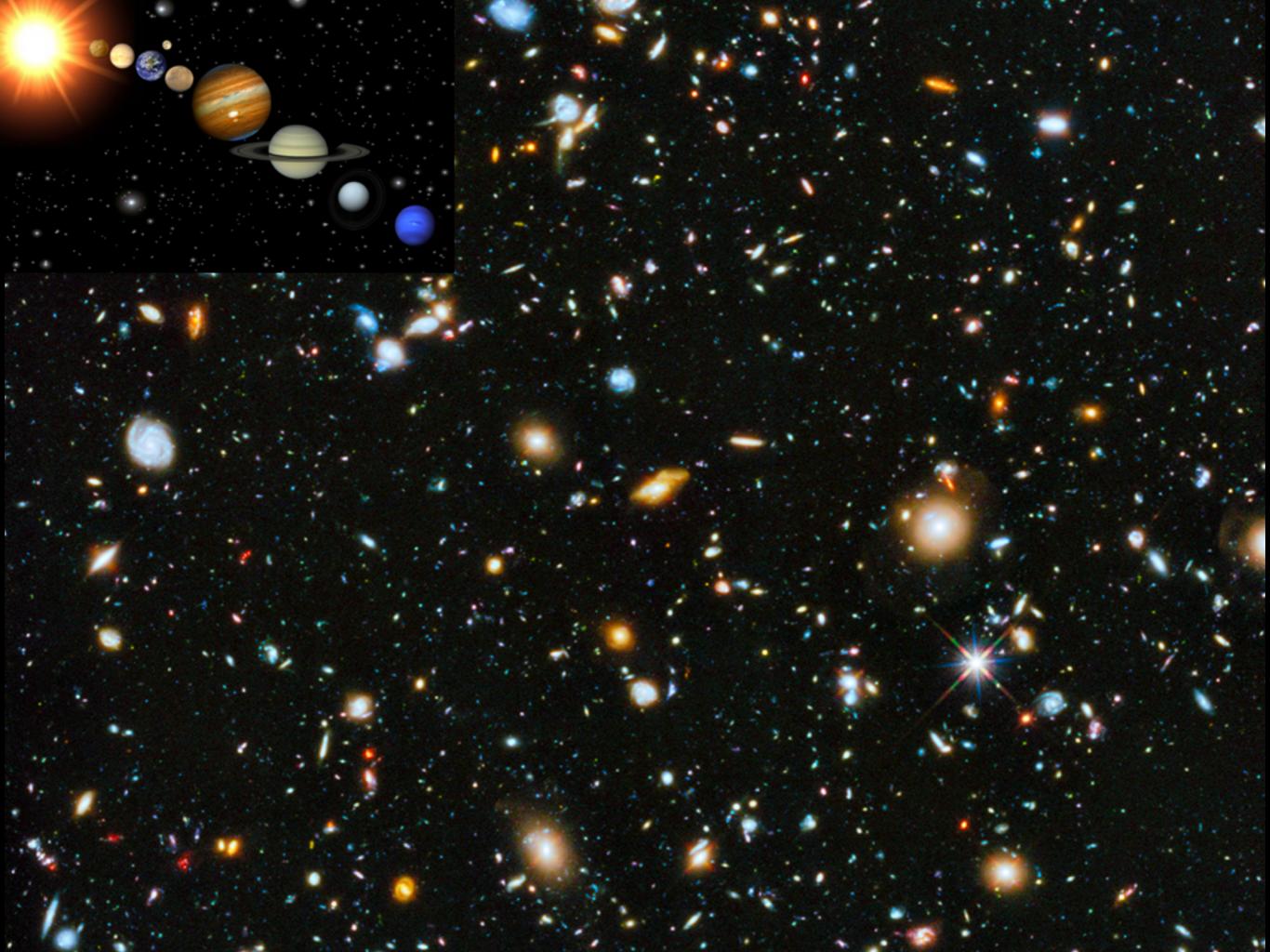


Old Newtonian Studies < 2017

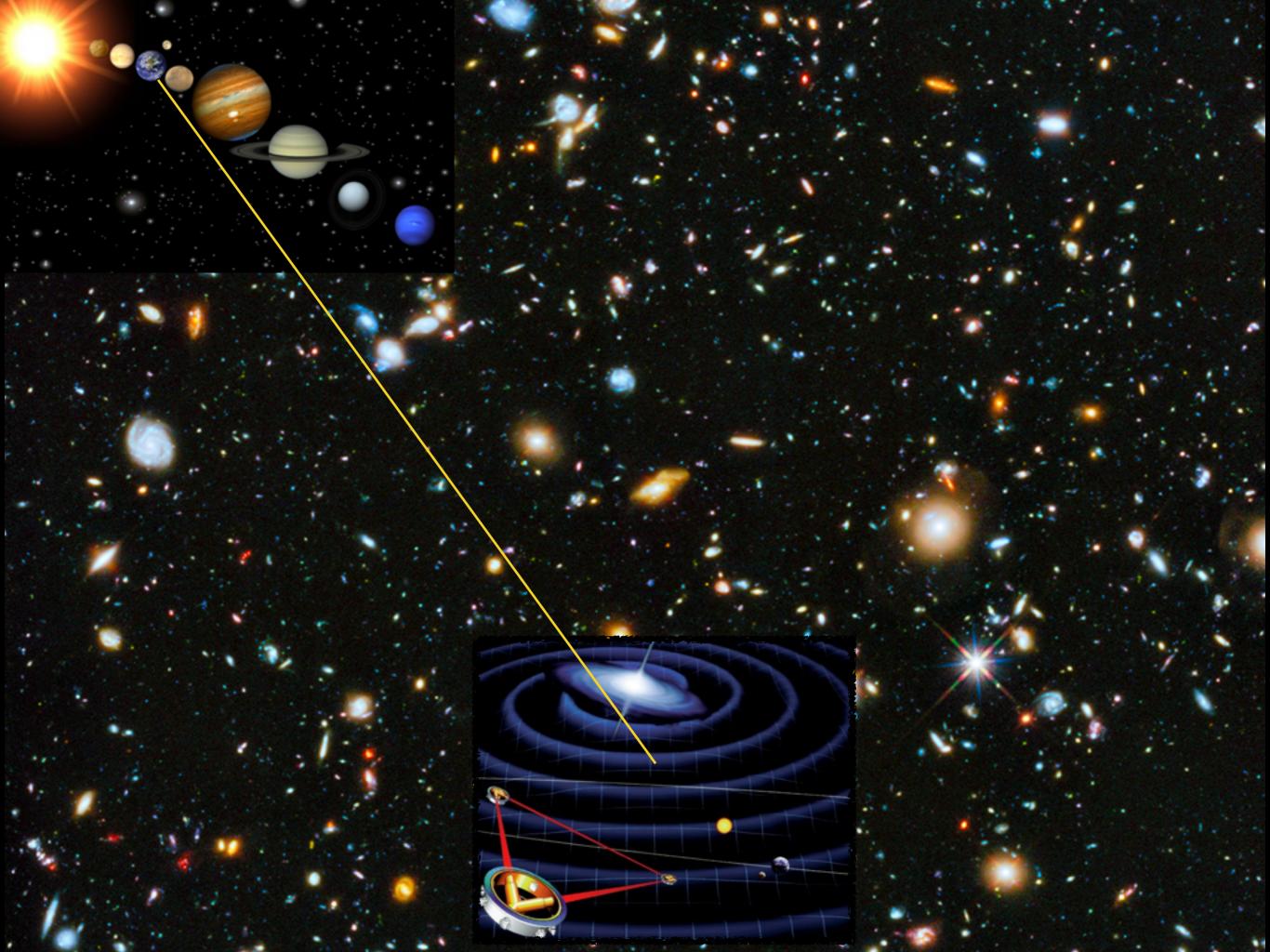


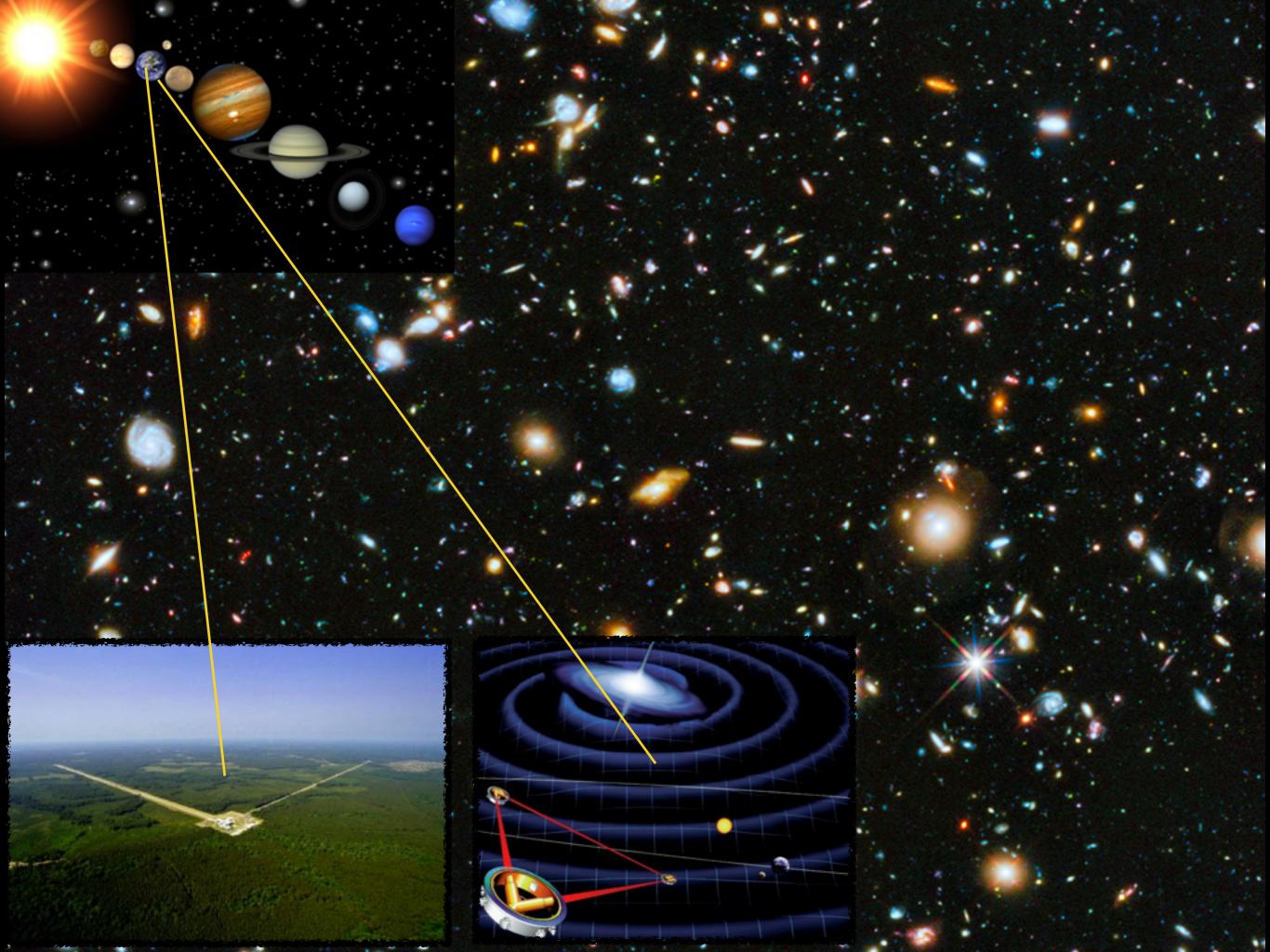
New Post-Newtonian Studies > 2017

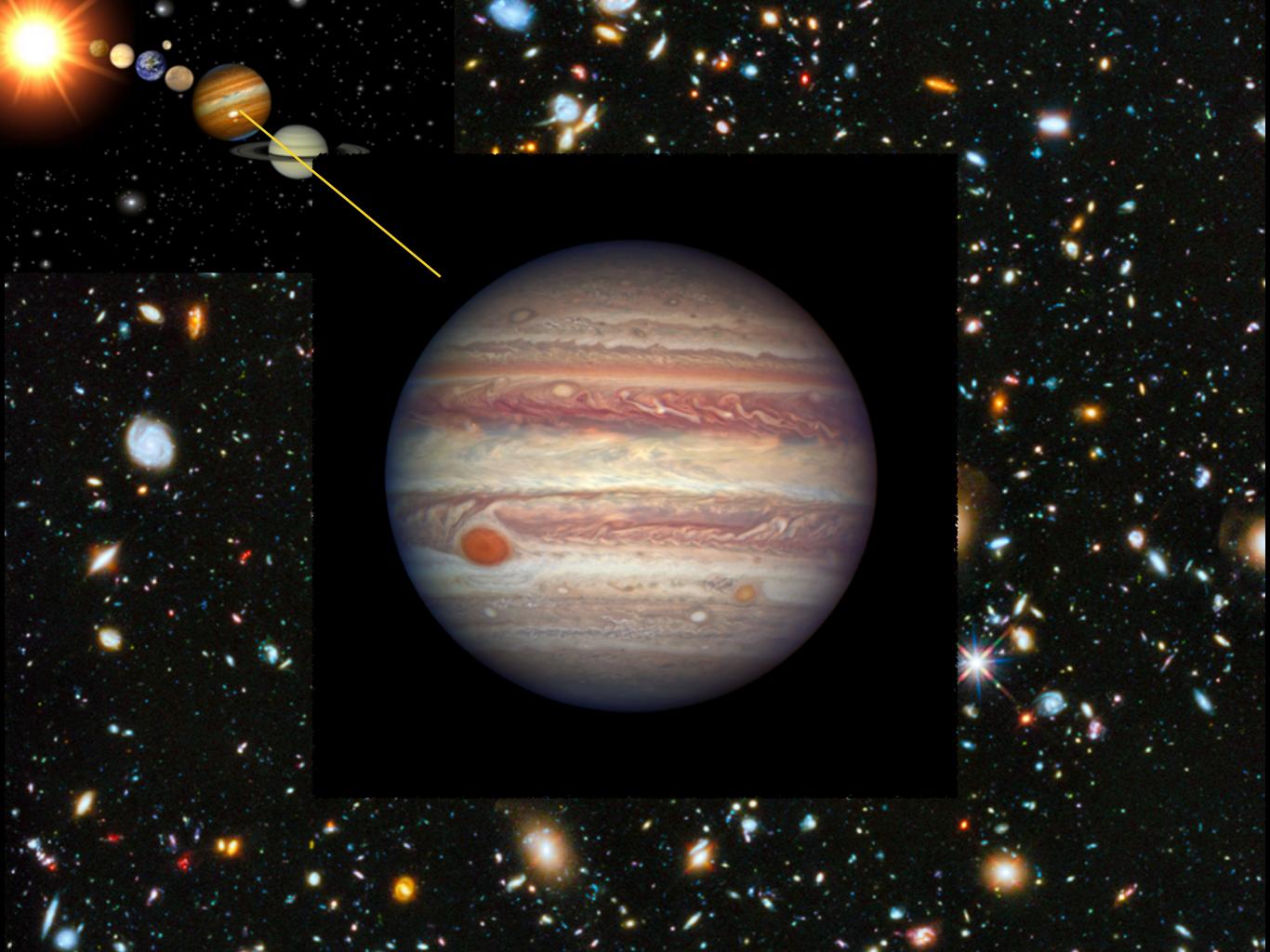


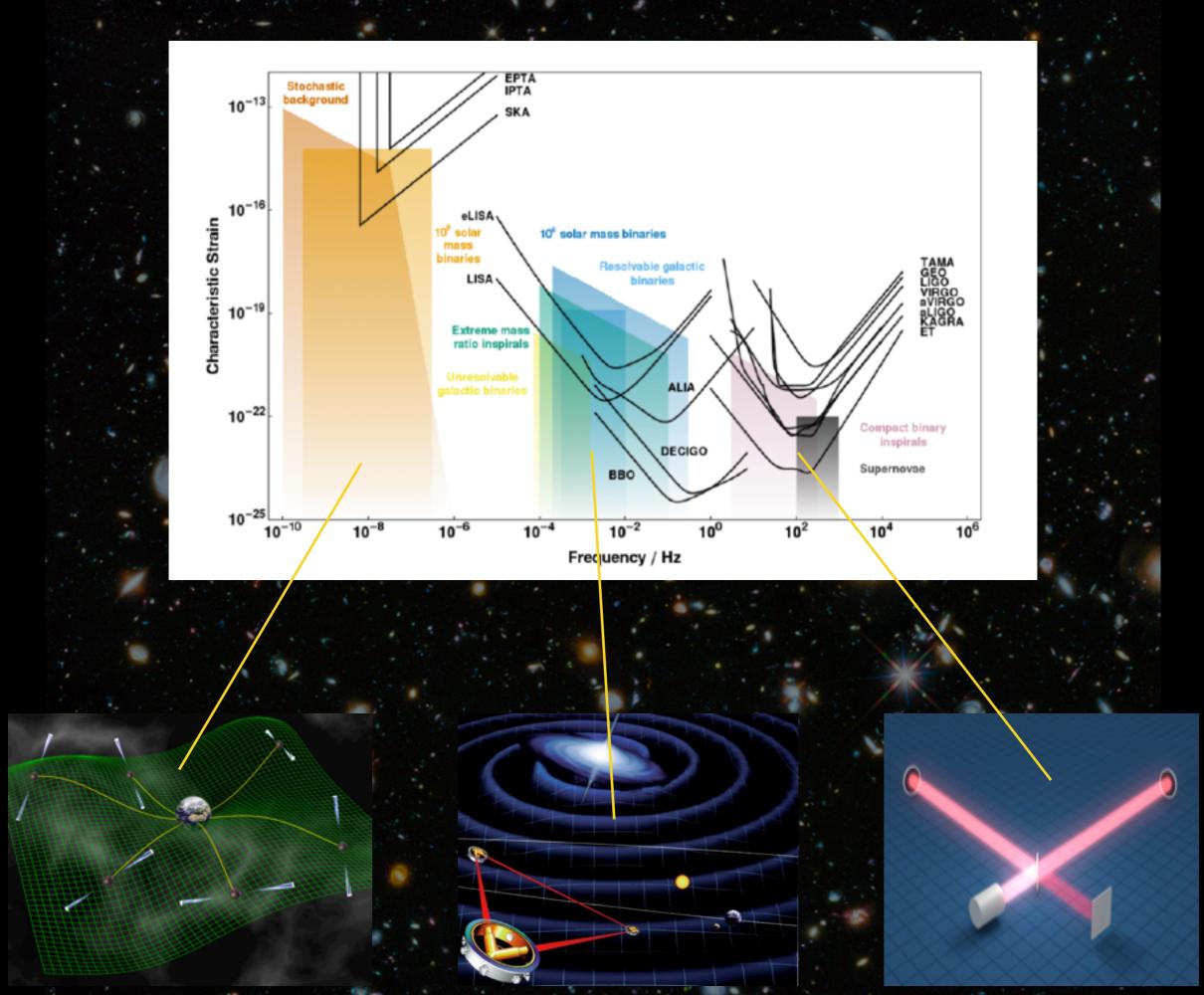




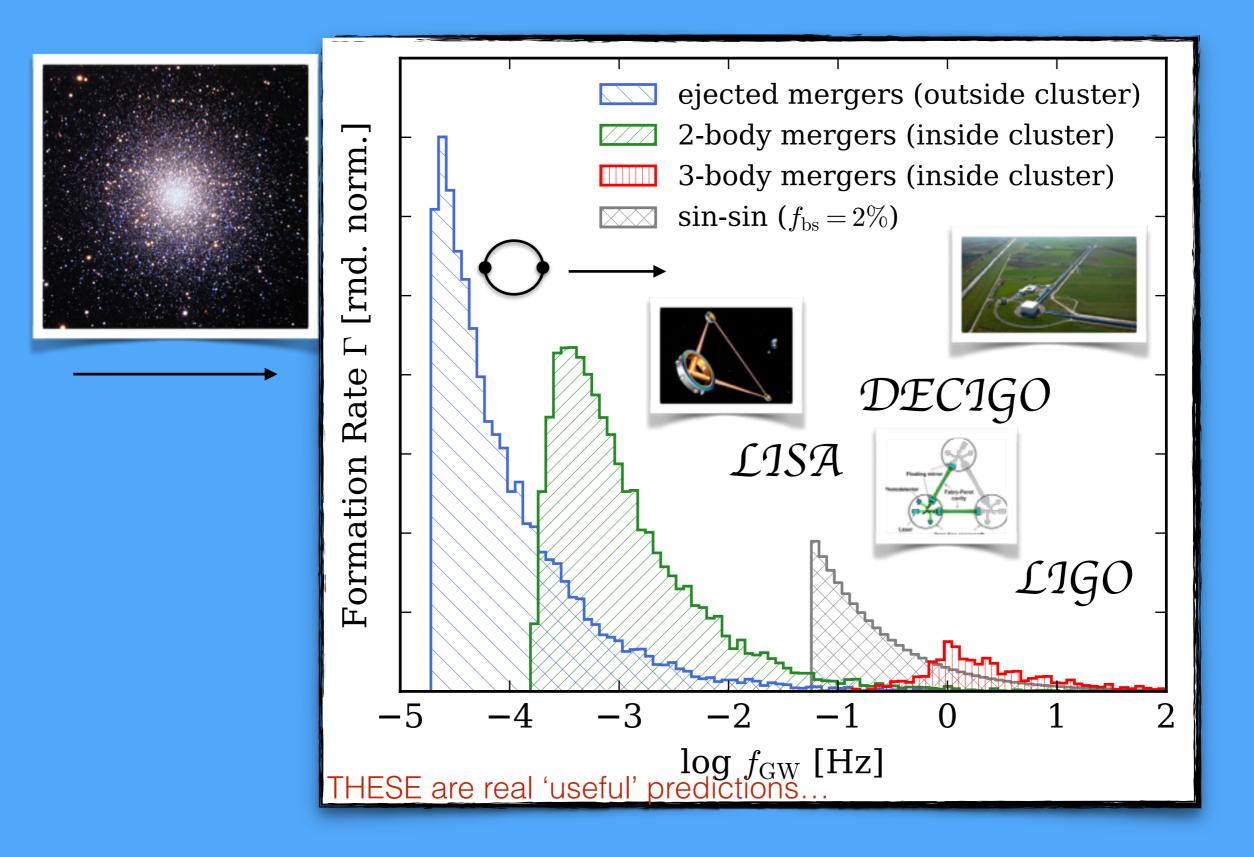








New Post-Newtonian Studies > 2017



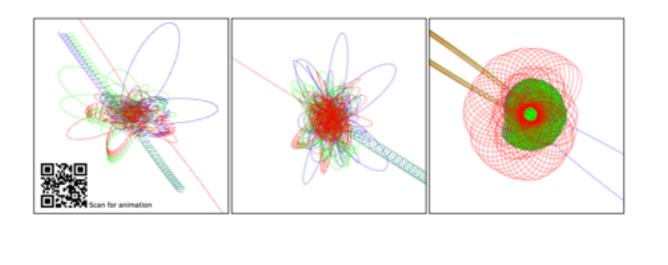
Background

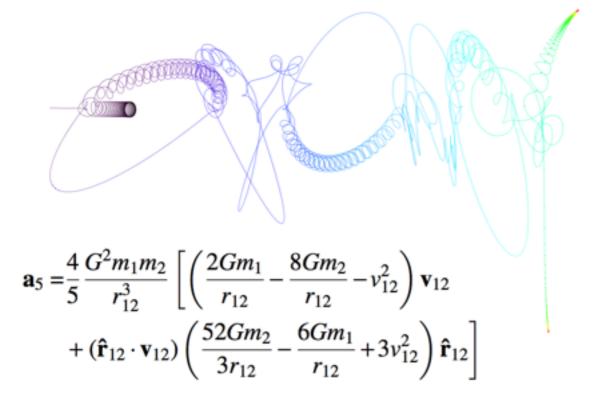
THE FORMATION OF ECCENTRIC COMPACT BINARY INSPIRALS AND THE ROLE OF GRAVITATIONAL WAVE EMISSION IN BINARY-SINGLE STELLAR ENCOUNTERS

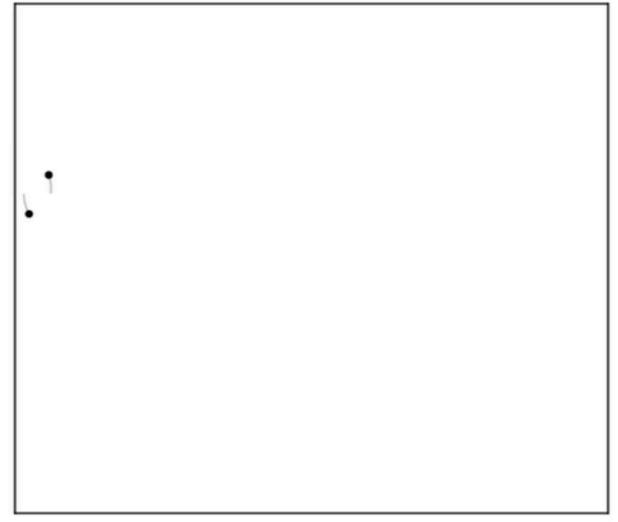
> JOHAN SAMSING¹, MORGAN MACLEOD², ENRICO RAMIREZ-RUIZ² Draft version October 29, 2018

ABSTRACT

The inspiral and merger of eccentric binaries leads to gravitational waveforms distinct from those generated by circularly merging binaries. Dynamical environments can assemble binaries with high eccentricity and peak frequencies within the *LIGO* band. In this paper, we study binary-single stellar scatterings occurring in dense







eccentric black hole mergers forming in globular clusters Samsing, 18.

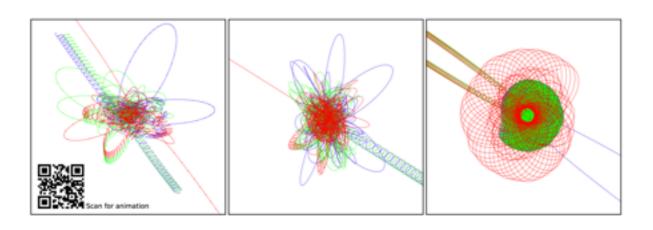
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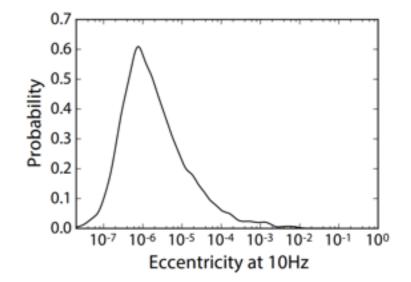


Binary Black Hole Mergers from Globular Clusters: Masses, Merger Rates, and the Impact of Stellar Evolution

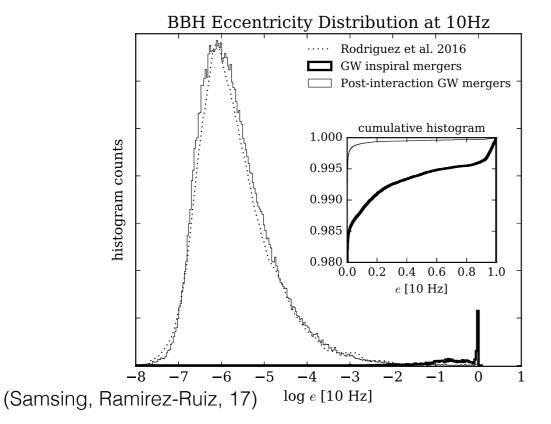
Carl L. Rodriguez,¹ Sourav Chatterjee,¹ and Frederic A. Rasio¹

¹Center for Interdisciplinary Exploration and Research in Astrophysics (CIERA) and Dept. of Physics and Astronomy, Northwestern University, 2145 Sheridan Rd, Evanston, IL 60208, USA (Dated: March 25, 2016)

The recent discovery of GW150914, the binary black hole merger detected by Advanced LIGO,







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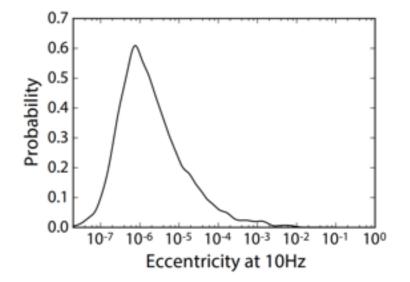
Why is it so difficult?

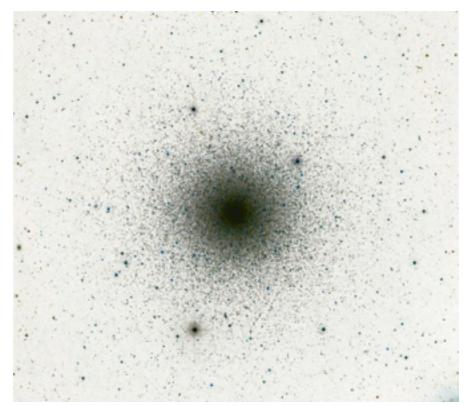
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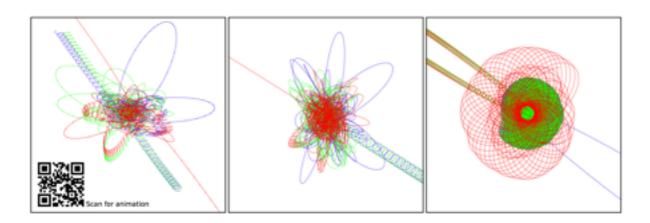
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Clusters give rise to predictable outcomes.

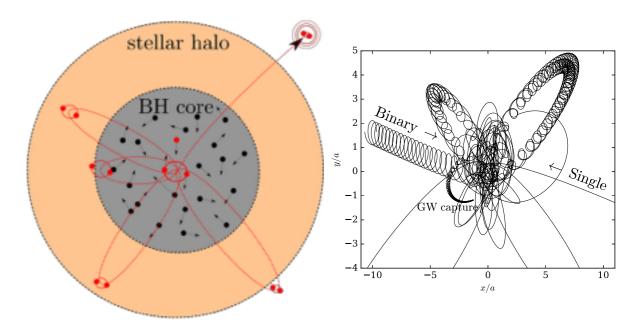
Pen and paper can reach percent precision!

Our two methods greatly complement each other!

Eccentric Black Hole Mergers Forming in Globular Clusters

Johan Samsing^{*} Department of Astrophysical Sciences, Princeton University, Peyton Hall, 4 Ivy Lane, Princeton, NJ 08544, USA.

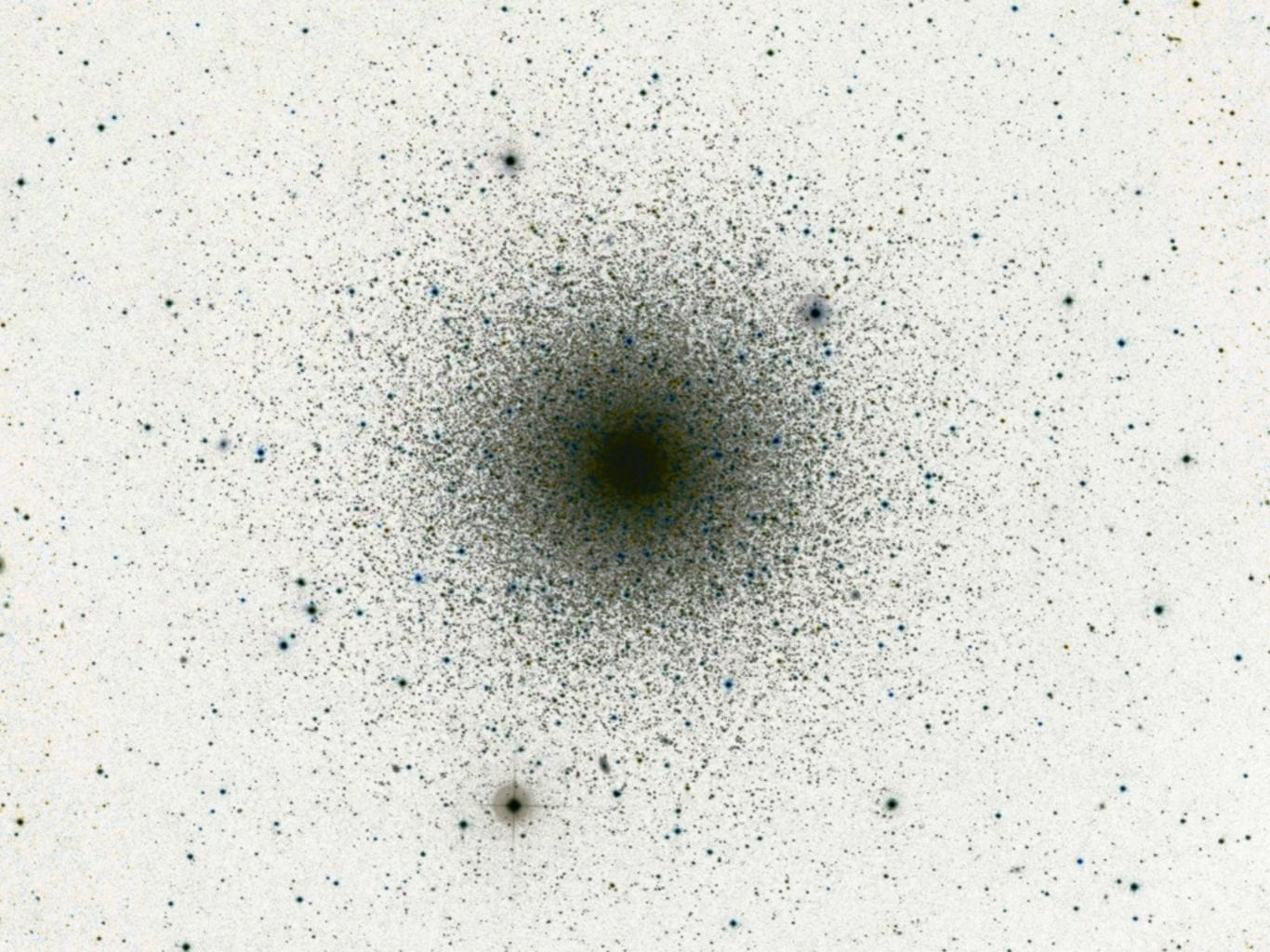
We derive the probability for a newly formed binary black hole (BBH) to undergo an eccentric gravitational wave (GW) merger during binary-single interactions inside a stellar cluster. By integrating over the hardening interactions such a BBH must undergo before ejection, we find that the observable rate of BBH mergers with eccentricity > 0.1 at 10 Hz relative to the rate of circular mergers and be which as $\sim 5\%$ for a typical globular cluster (GC). This further suggests that BBH mergers forming throug GW captures in binary-single interactions, eccentric or not, are likely to constitute $\sim 10\%$ of the total BBH merger rate from GCs. Such GW capture mergers can only



Post-Newtonian Dynamics in Dense Star Clusters: Formation, Masses, and Merger Rates of Highly-Eccentric Black Hole Binaries

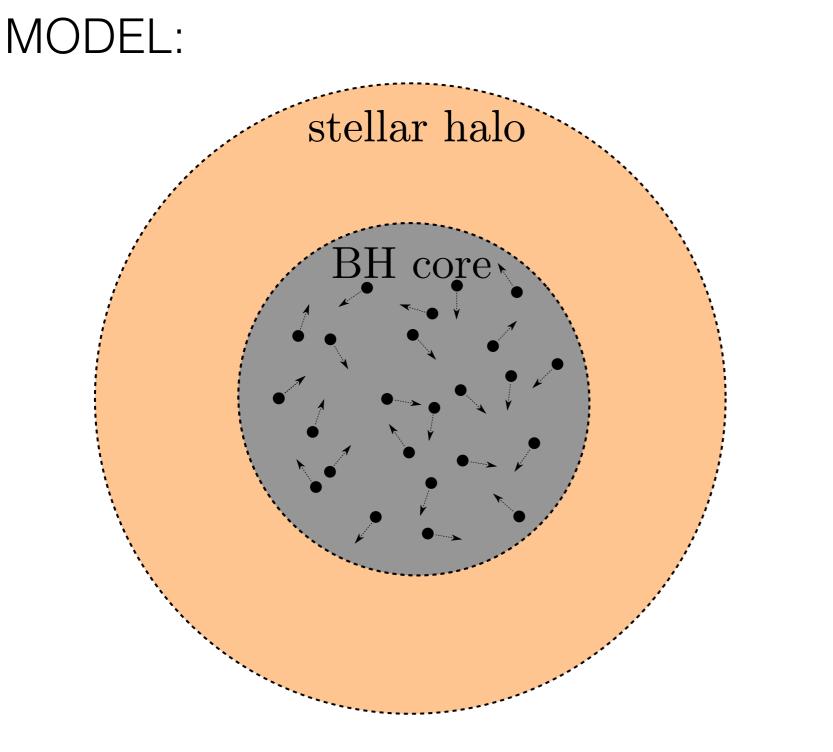
Carl L. Rodriguez,¹ Pau Amaro-Seoane,² Sourav Chatterjee,³ Kyle Kremer,⁴ Frederic A. Rasio,⁴ Johan Samsing,⁵ Claire S. Ye,⁴ and Michael Zevin⁴

Using state-of-the-art dynamical simulations of globular clusters, including radiation reaction during black hole encounters and a cosmological model of star cluster formation, we create a realistic population of dynamically-formed binary black hole mergers across cosmic space and time. We show that in the local universe, 10% of these binaries form as the result of gravitational-wave emission between unbound black holes during claotic resonant encounters, with roughly half of those events having eccentricities detect black current ground-based gravitational-wave detectors. The mergers



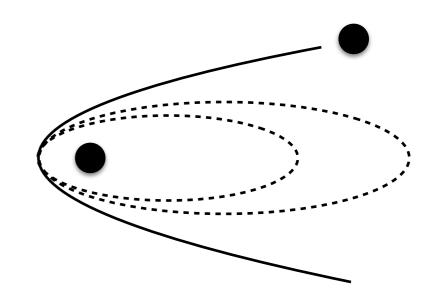
stellar halo

BH core



 $f\approx \frac{1}{\pi}\sqrt{\frac{2Gm}{r_f^3}}$

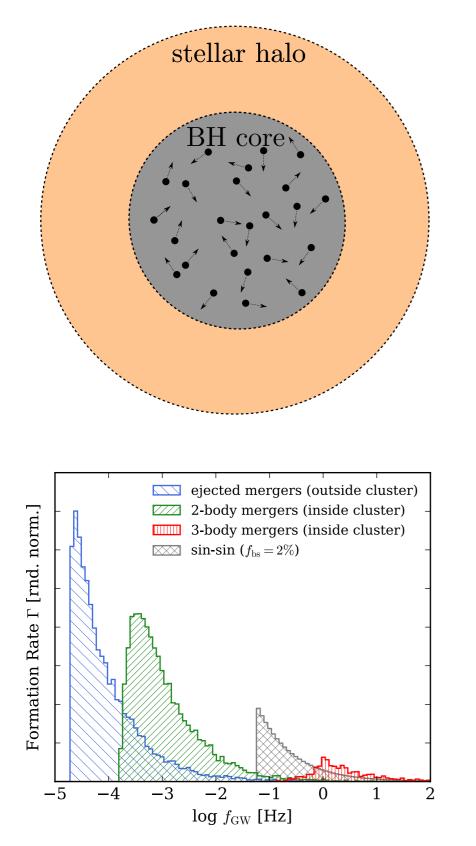
 $r_f \approx \left(\frac{2Gm}{f^2\pi^2}\right)^{1/3}$

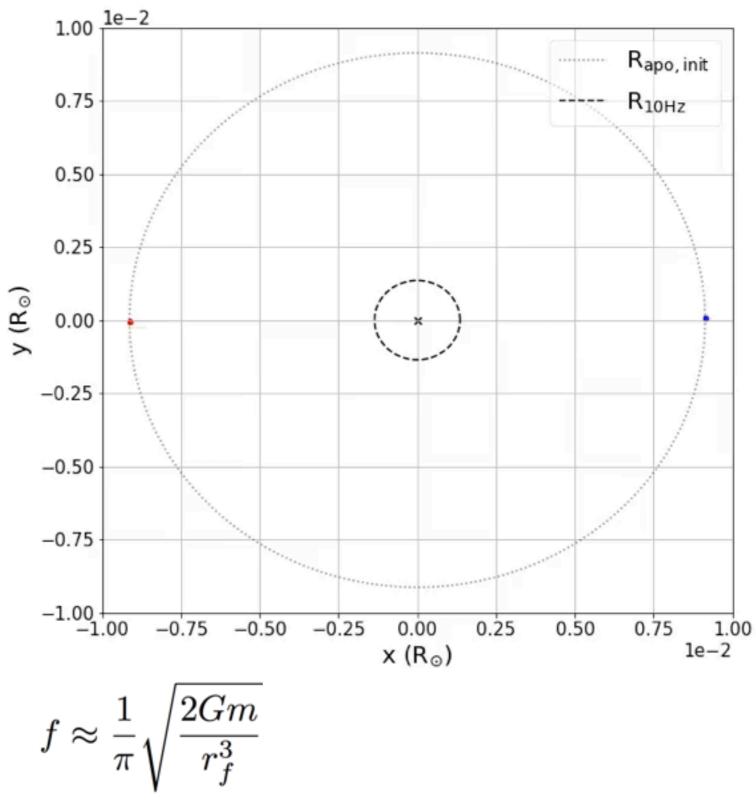


How do BBHs form and merge?

What is the peak freq. dist? Correlate this with e!

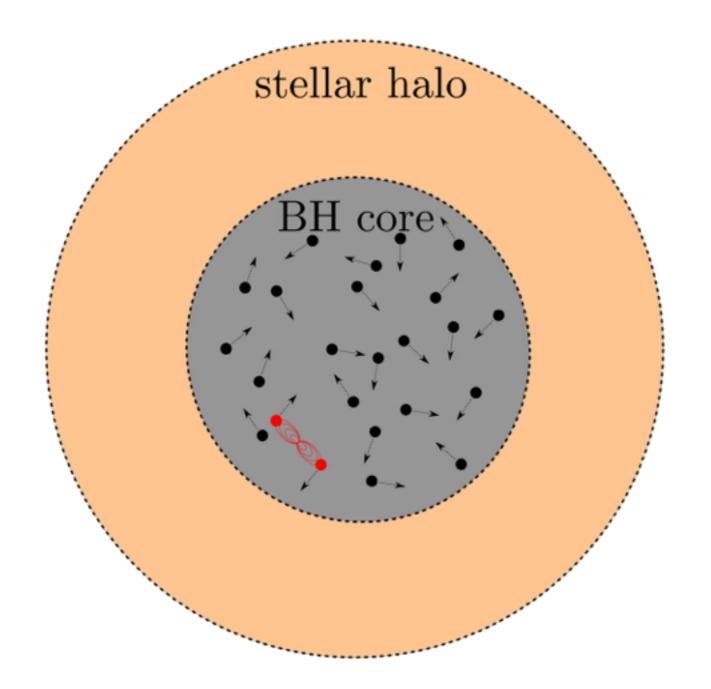
MODEL:

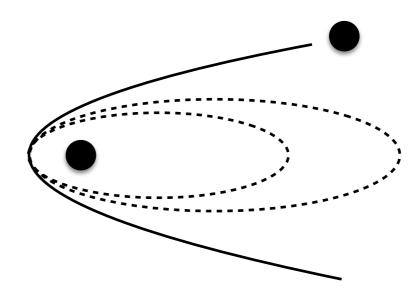




Merger Type: Single-Single

Capture:

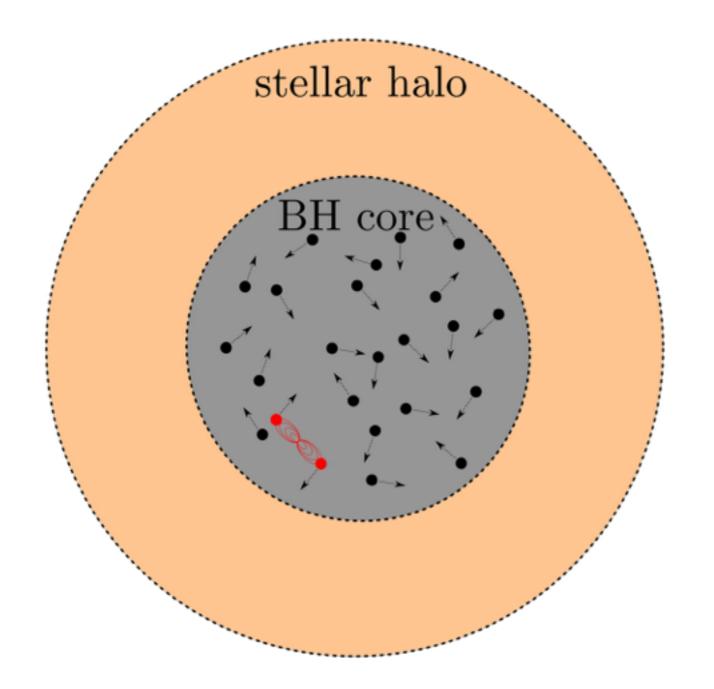




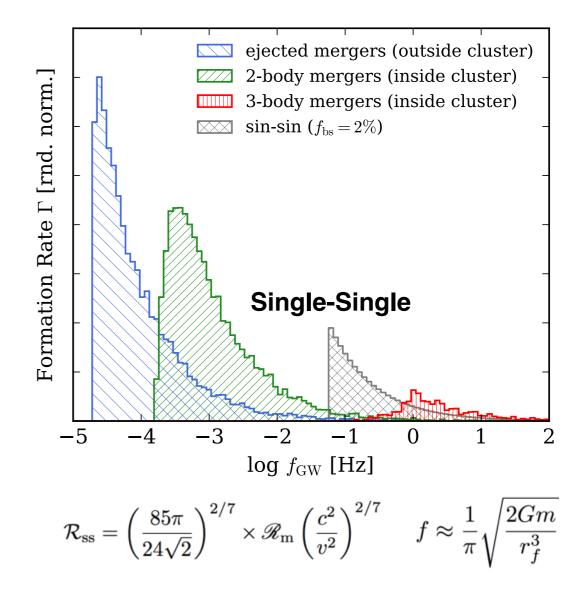
 $\Delta E_{\rm p} \approx (85\pi/12)G^{7/2}c^{-5}m^{9/2}r_{\rm p}^{-7/2}$ $E_{\rm ss} \approx \mu v^2/2$

$$\mathcal{R}_{\rm ss} = \left(\frac{85\pi}{24\sqrt{2}}\right)^{2/7} \times \mathscr{R}_{\rm m} \left(\frac{c^2}{v^2}\right)^{2/7}$$

Merger Type: Single-Single



S-S captures do not only operate in Galactic Nuclei!



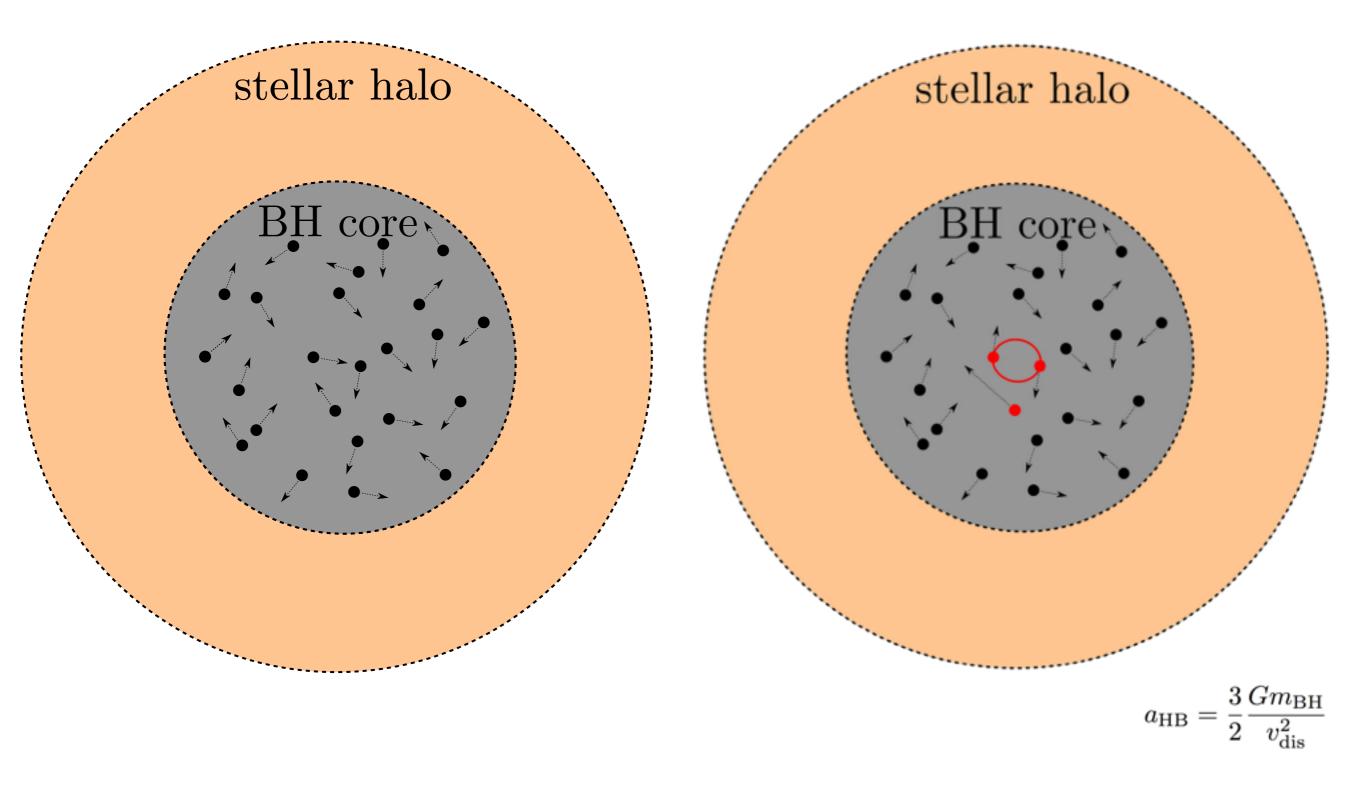
Distribution depends on:

BH distribution (Plummer/Uniform..) Binary Fraction

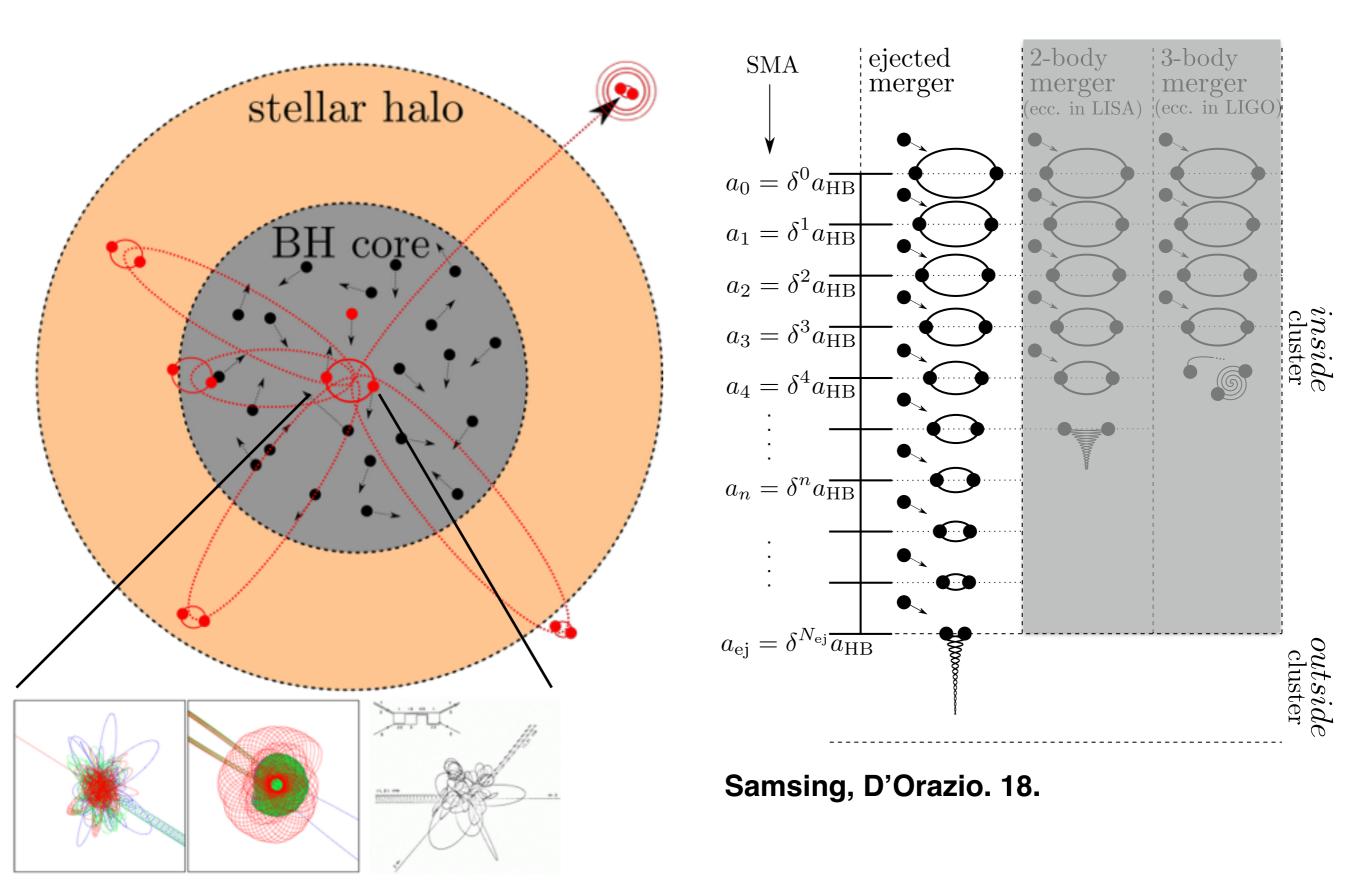
see our new paper:

'GW captures of single BHs in GCs' (Samsing, et al. 2019)

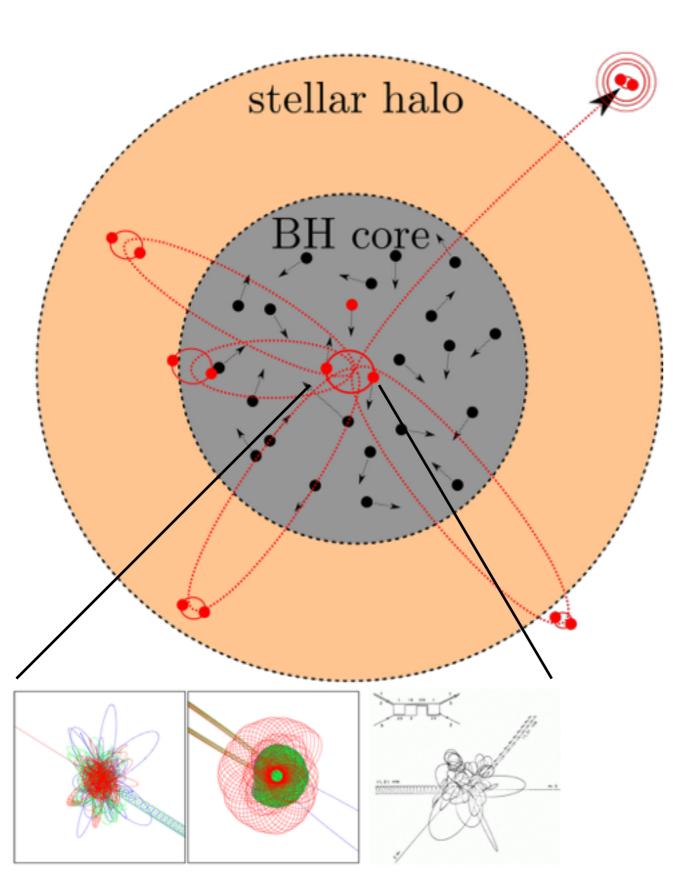
Few-body BBH mergers: Formation of a BBH

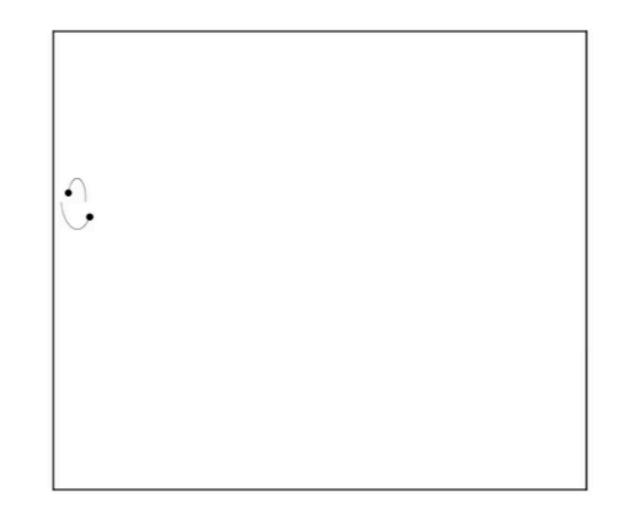


Merger Type: Ejected Merger

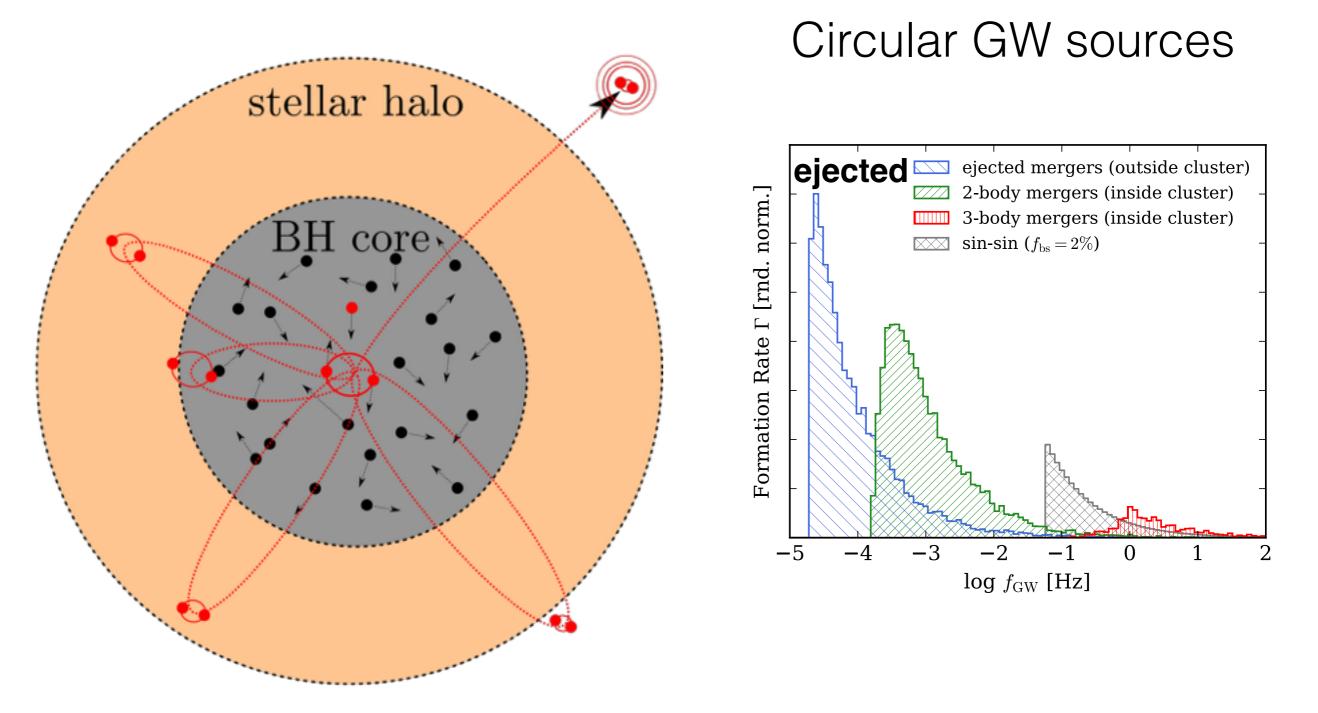


Merger Type: Ejected Merger

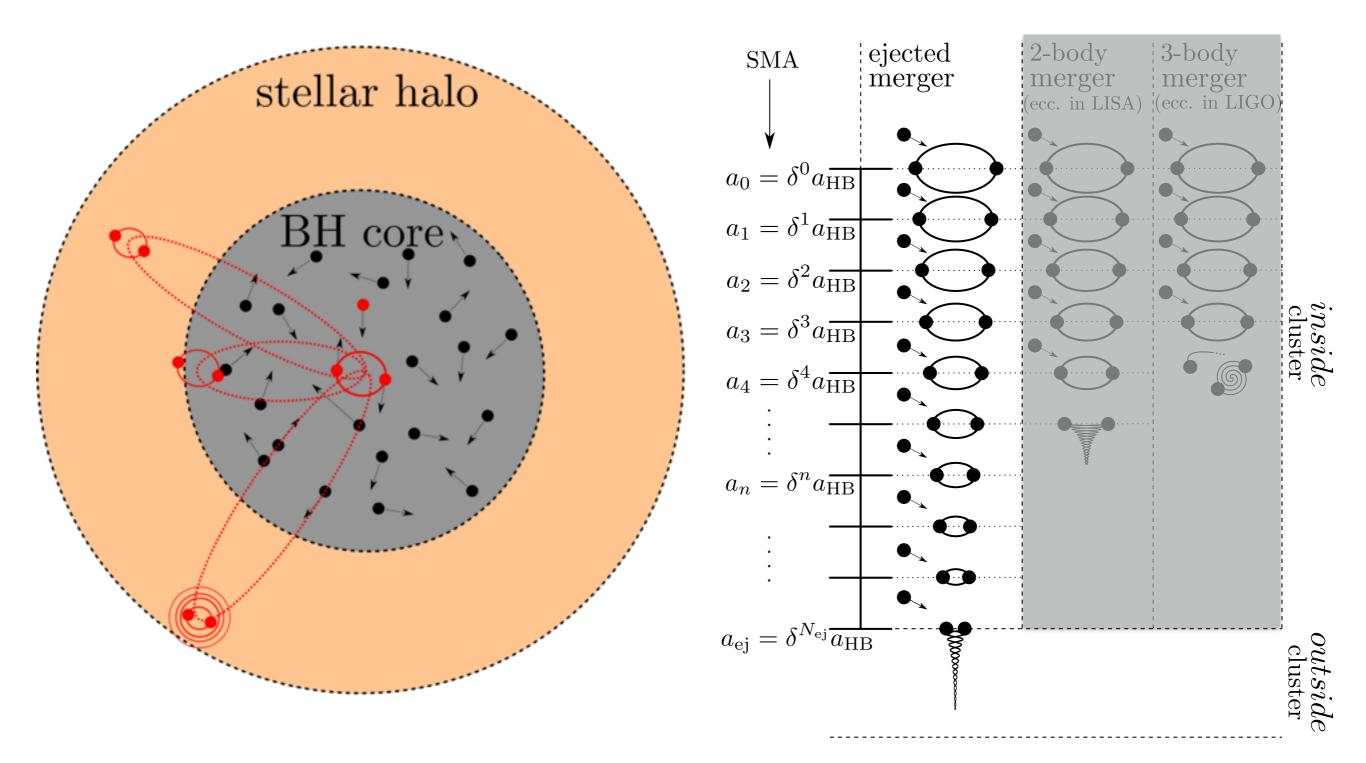


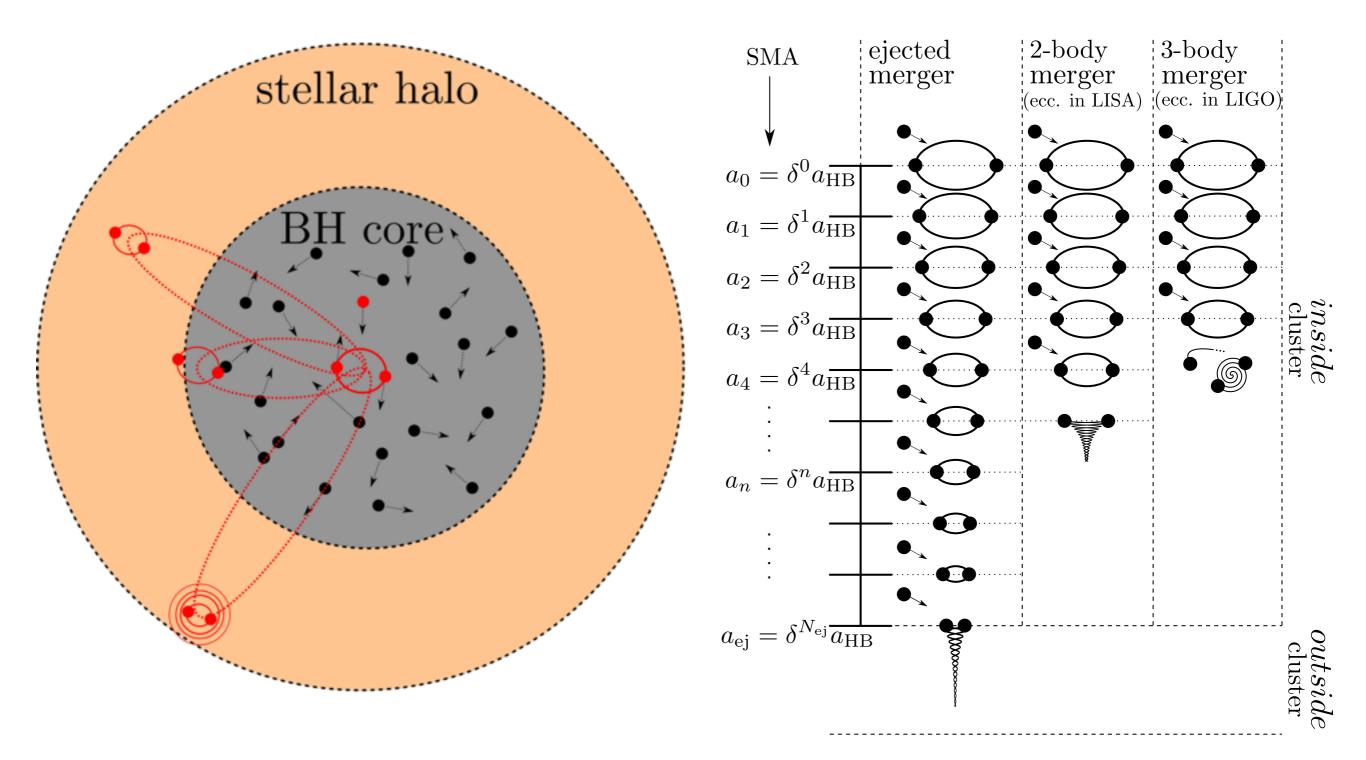


Merger Type: Ejected Merger

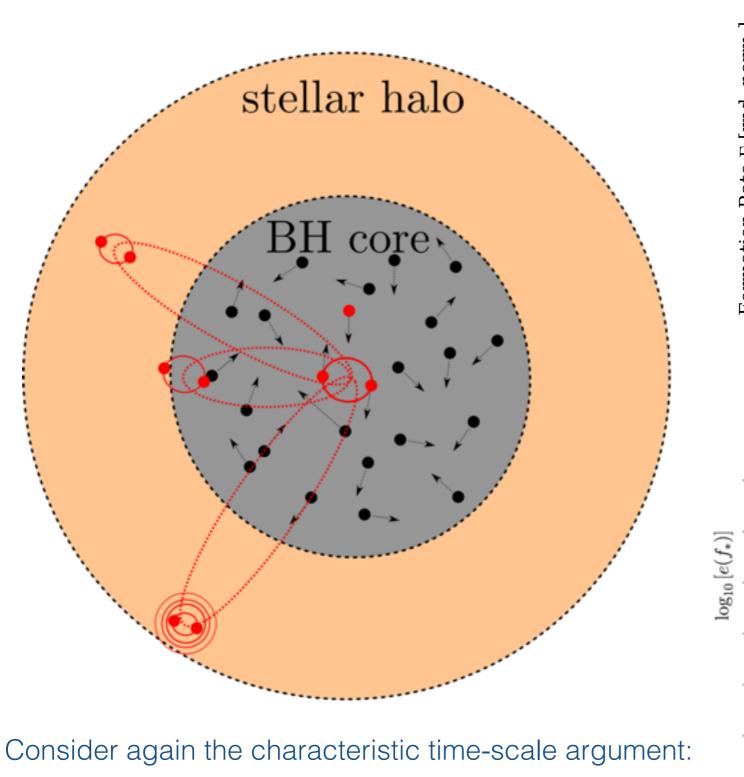


Important point: It is all about characteristic time scales! $f_{r,0}^{\text{peak}}(\mathcal{T}) \approx 2 \cdot 10^{-5} \text{ Hz} \left(\frac{\mathcal{T}}{10^{10} \text{ yrs}}\right)^{-3/7} \left(\frac{a}{0.5 \text{ au}}\right)^{3/14} \left(\frac{m}{30 M_{\odot}}\right)^{-11/14}$

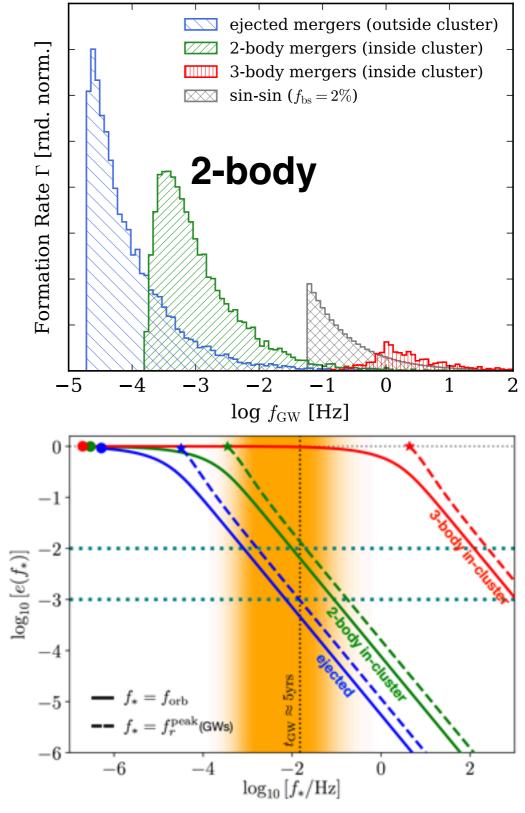




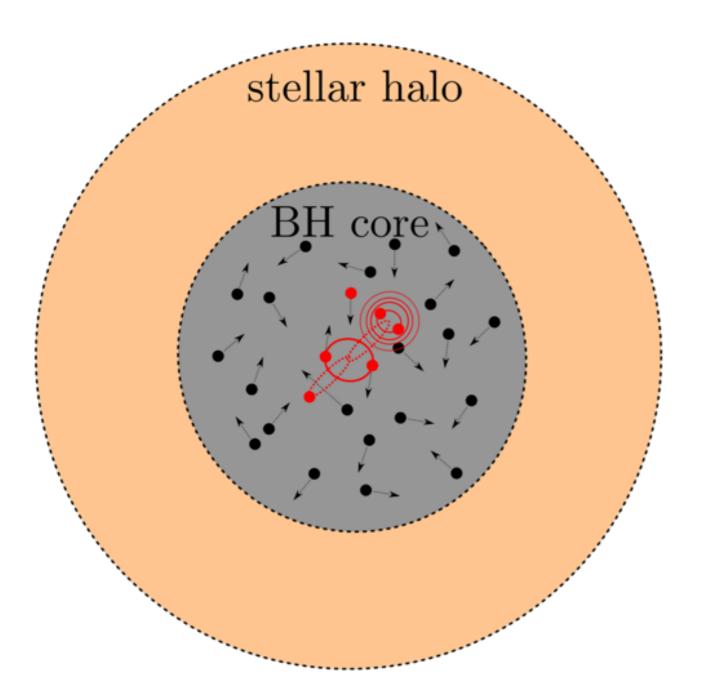
Eccentric LISA sources



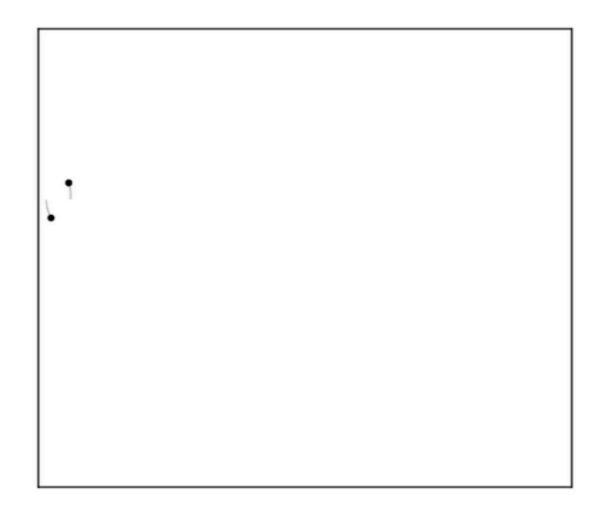
$$f_{r,0}^{\text{peak}}(\mathcal{T}) \approx 2 \cdot 10^{-5} \text{ Hz} \left(\frac{\mathcal{T}}{10^{10} \text{ yrs}}\right)^{-3/7} \left(\frac{a}{0.5 \text{ au}}\right)^{3/14} \left(\frac{m}{30 M_{\odot}}\right)^{-11/14}$$



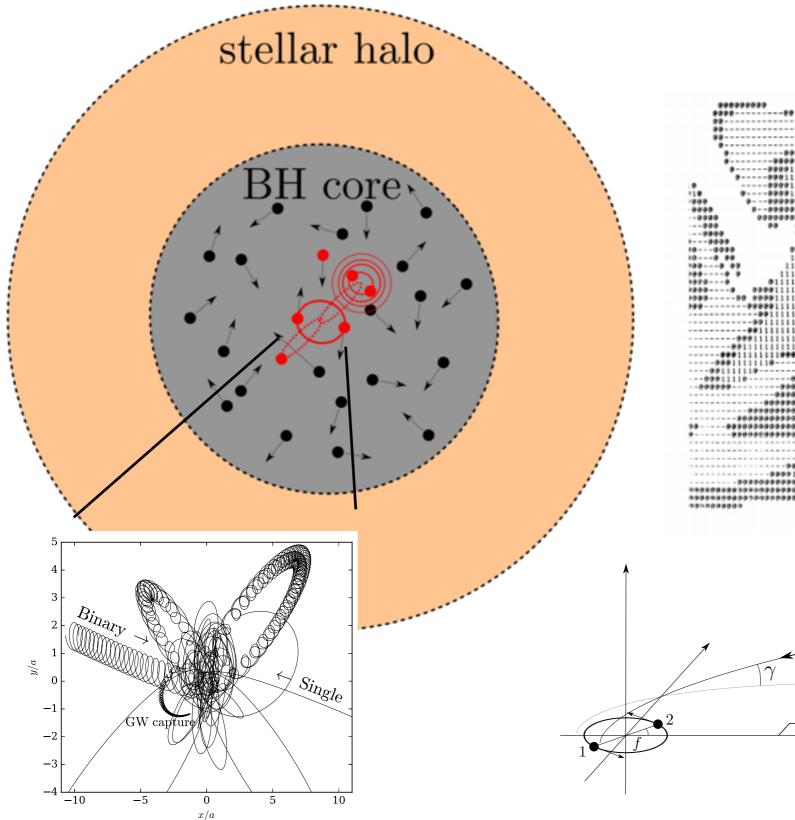
series of papers: Samsing, D'Orazio

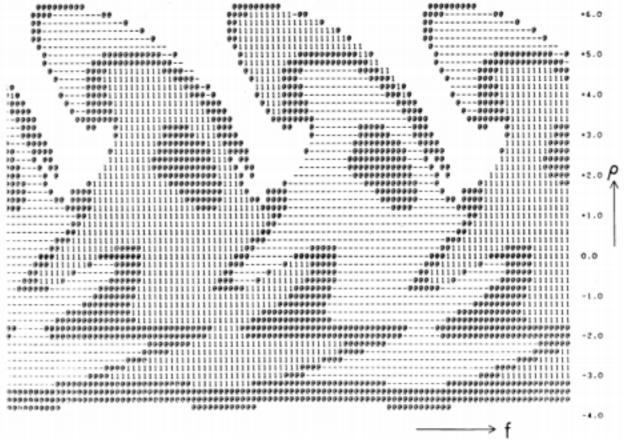


Post-Newtonian



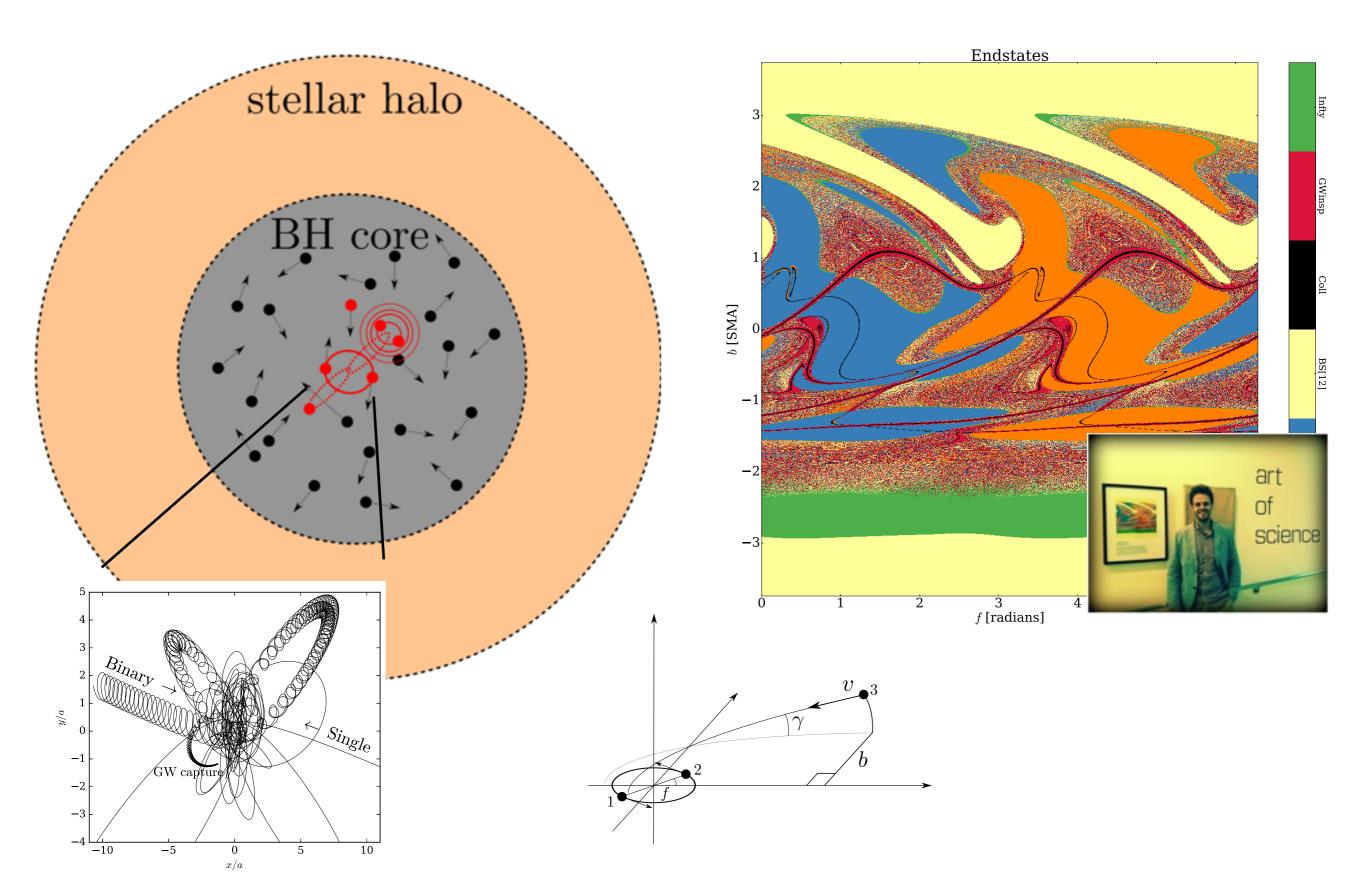


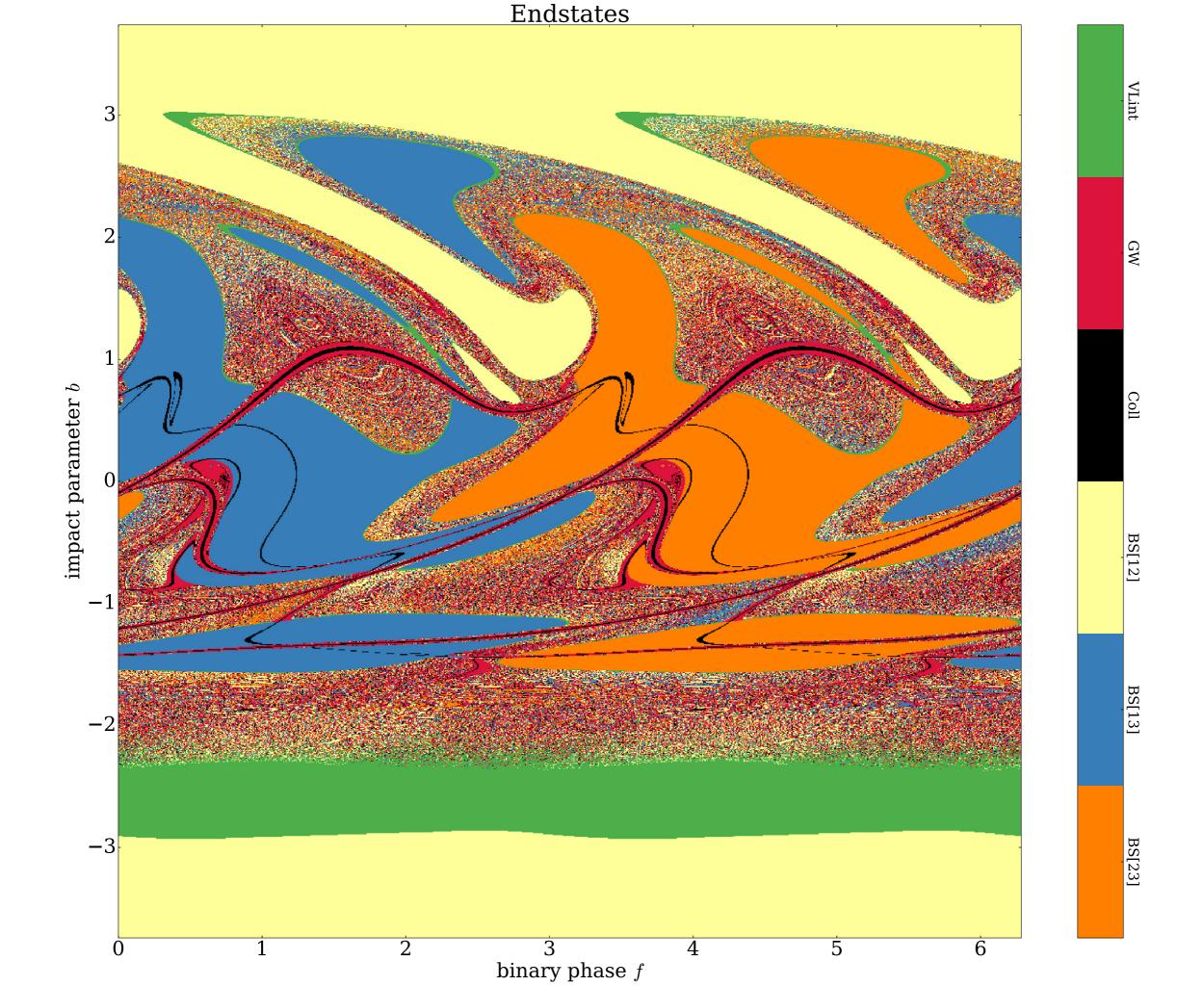




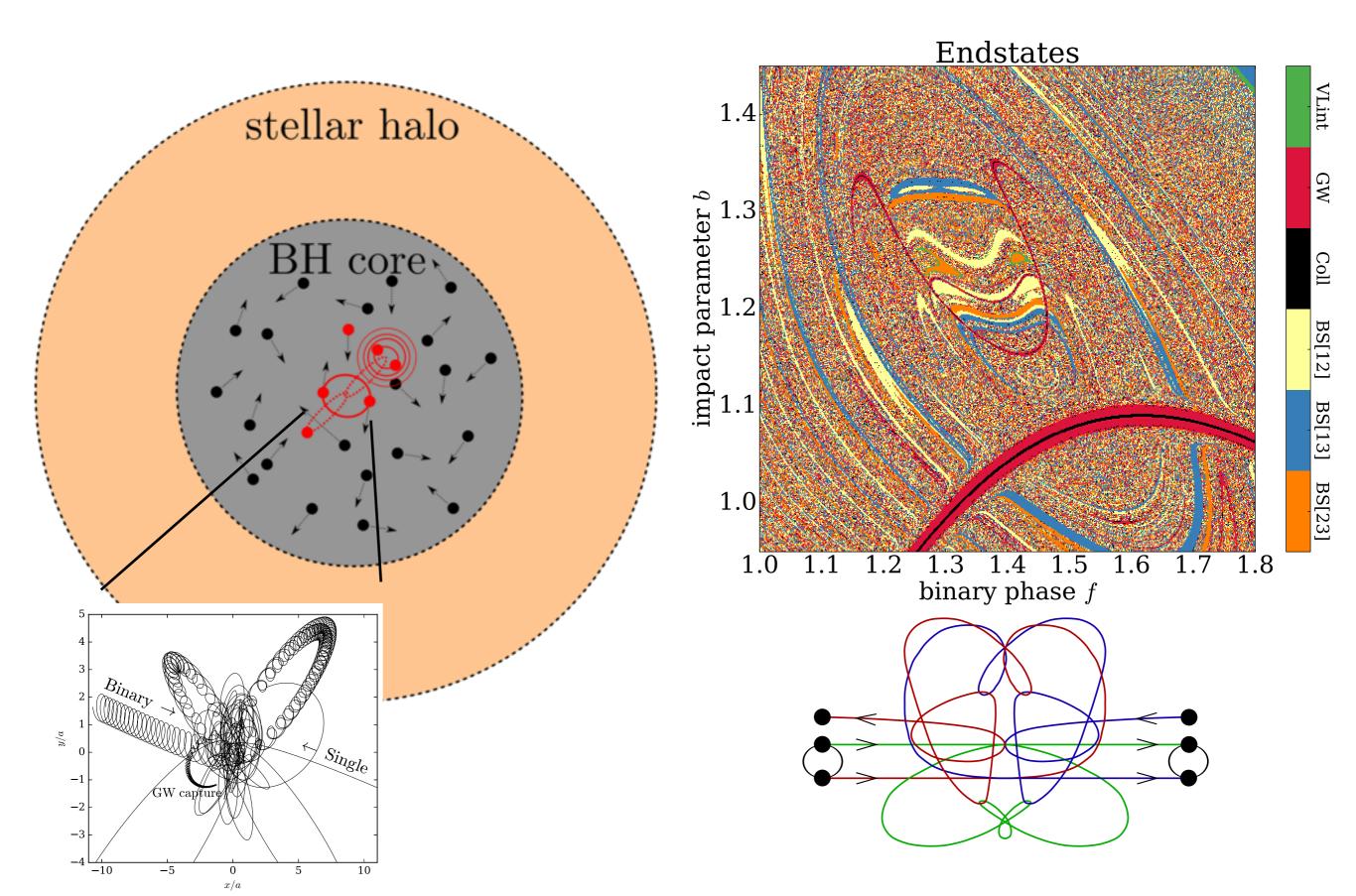
 \boldsymbol{v}

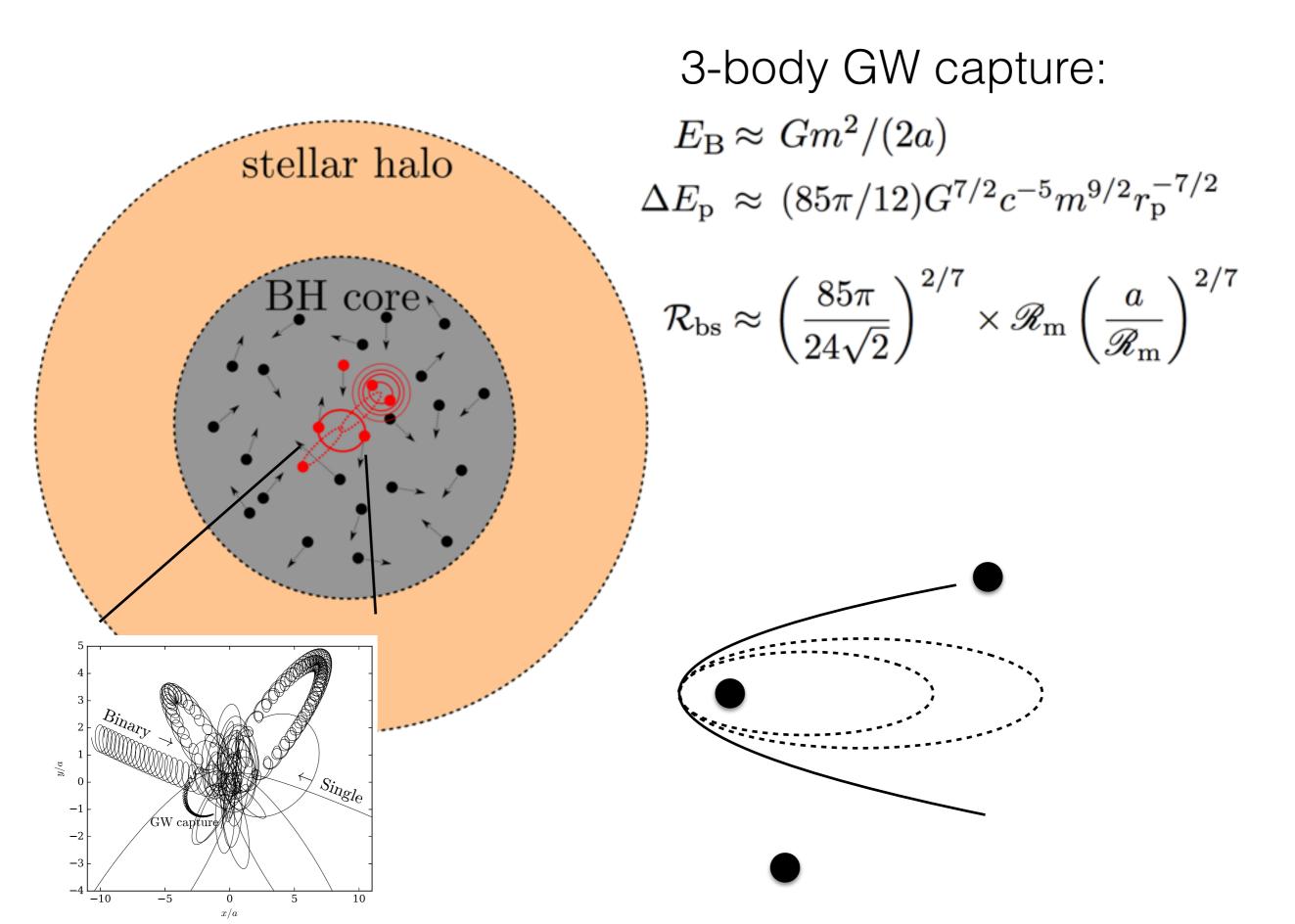
Topology

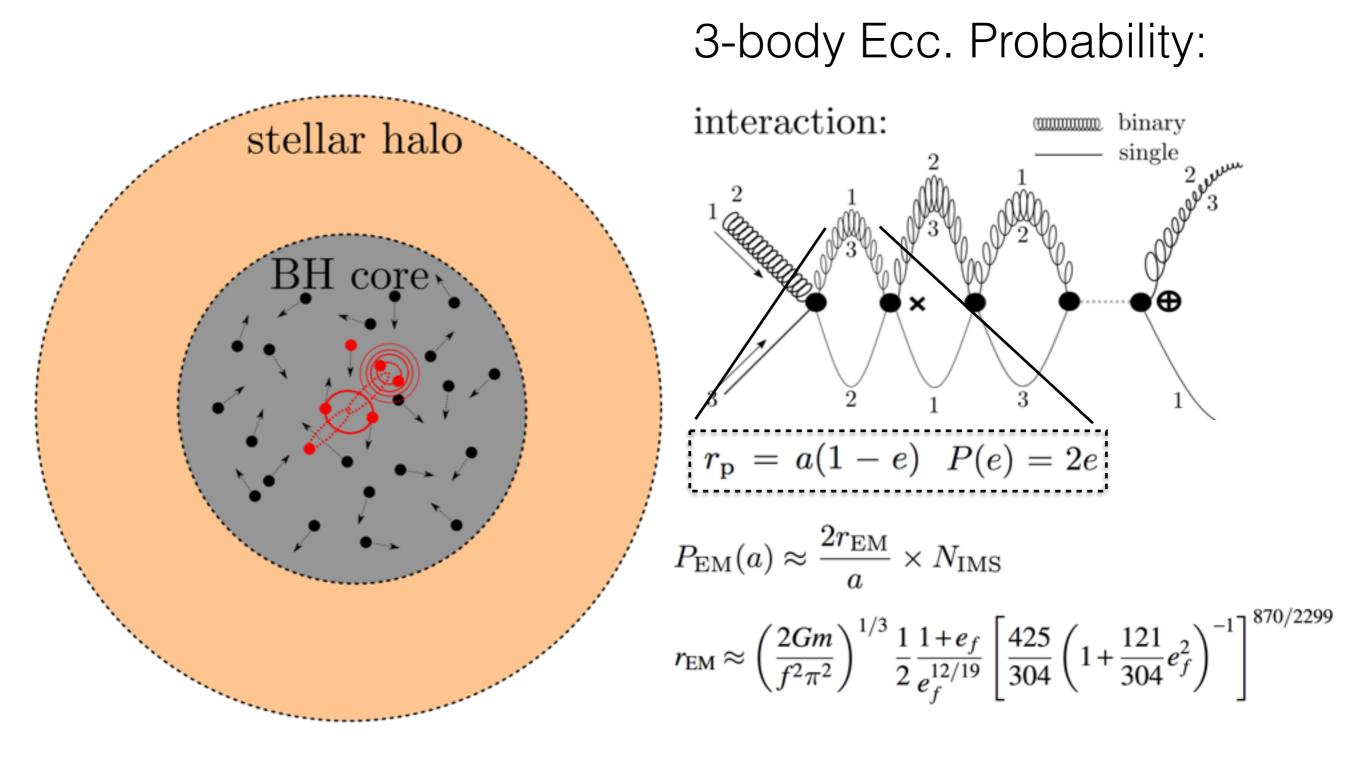




Topology



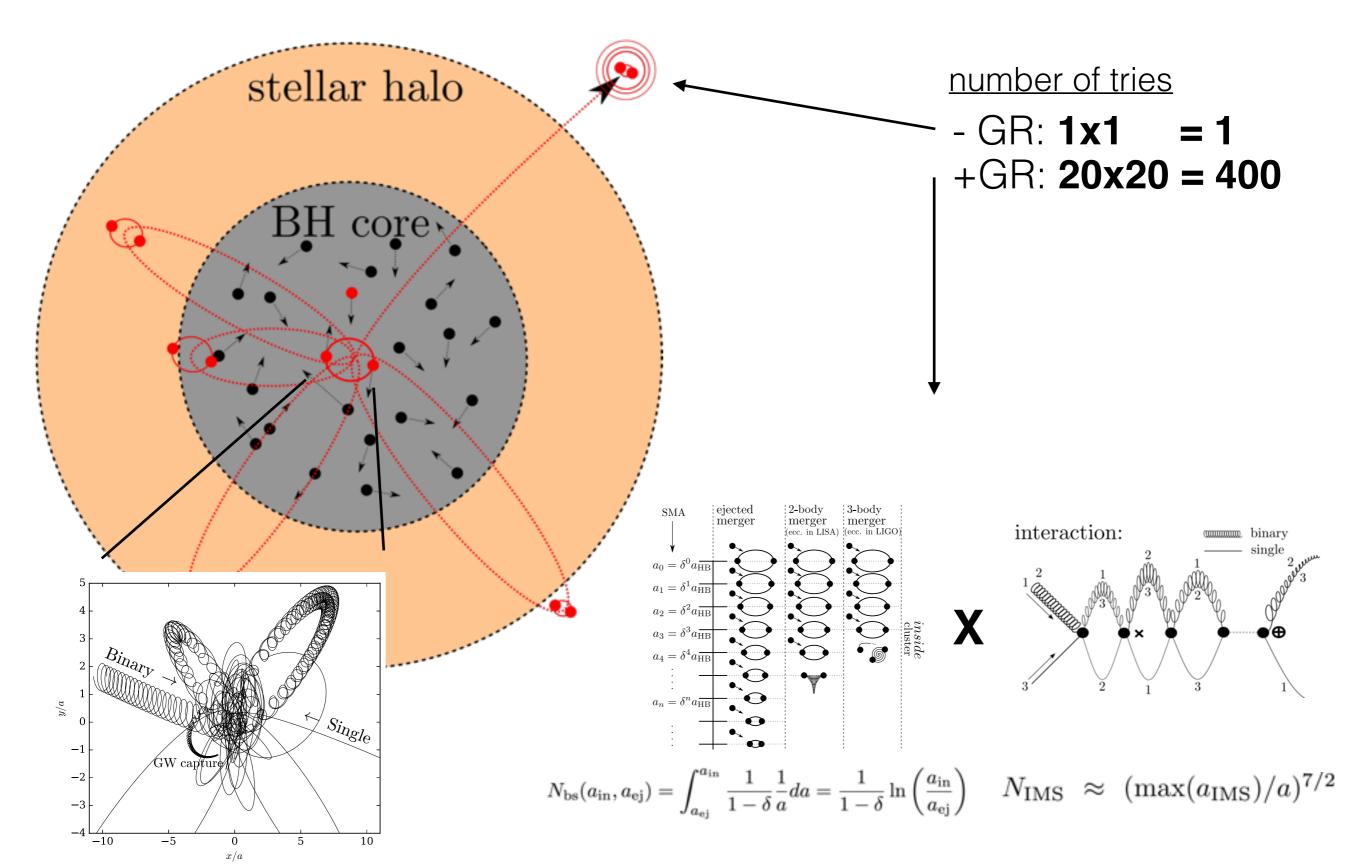




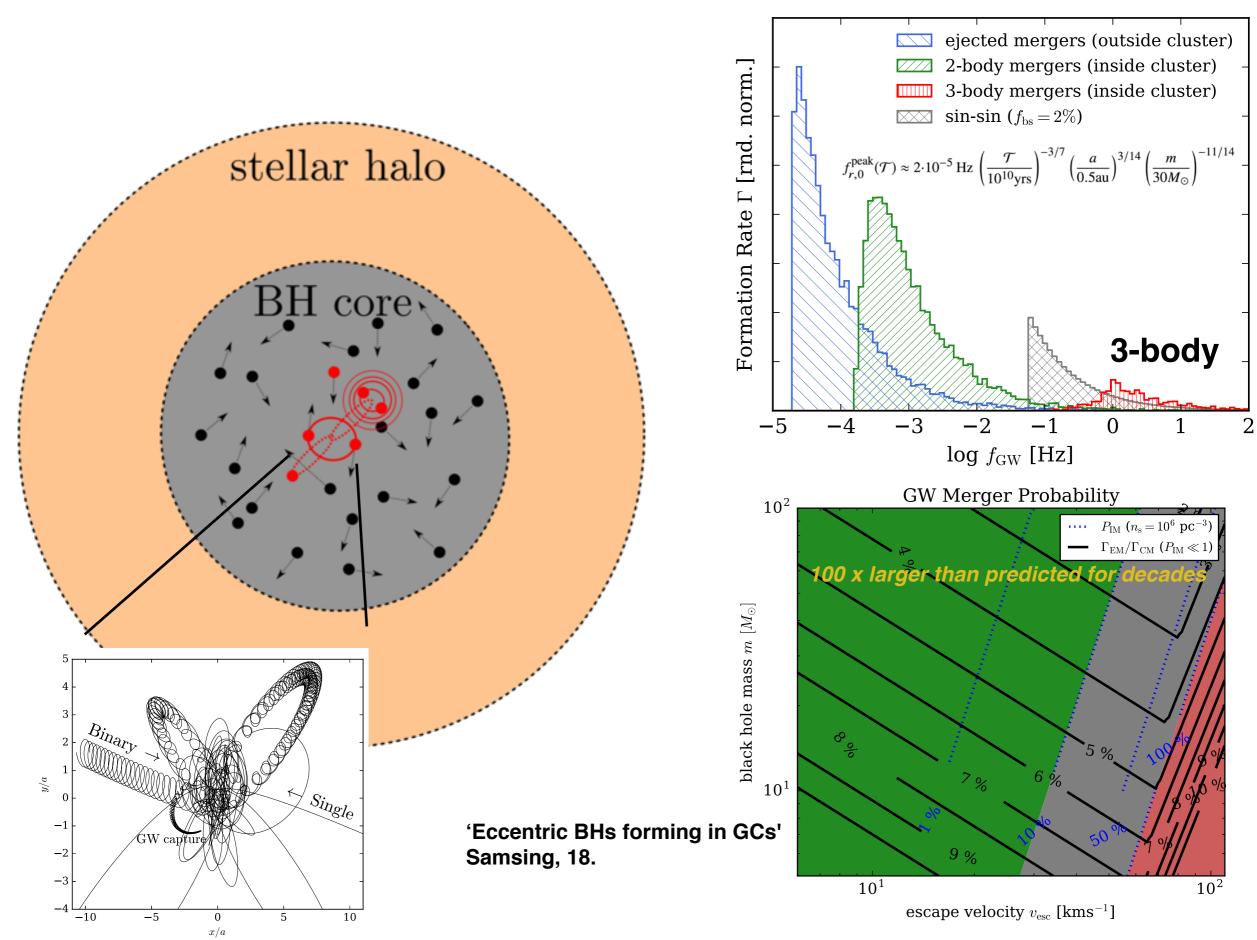
$$P_{\rm EM}(a_{\rm in}, a_{\rm ej}) = \frac{1}{1-\delta} \int_{a_{\rm ej}}^{a_{\rm in}} \frac{P_{\rm EM}(a)}{a} da \approx \frac{P_{\rm EM}(a_{\rm ej})}{1-\delta}$$

Eccentric Black Hole Mergers Forming in Globular Clusters *Authors: Johan Samsing*

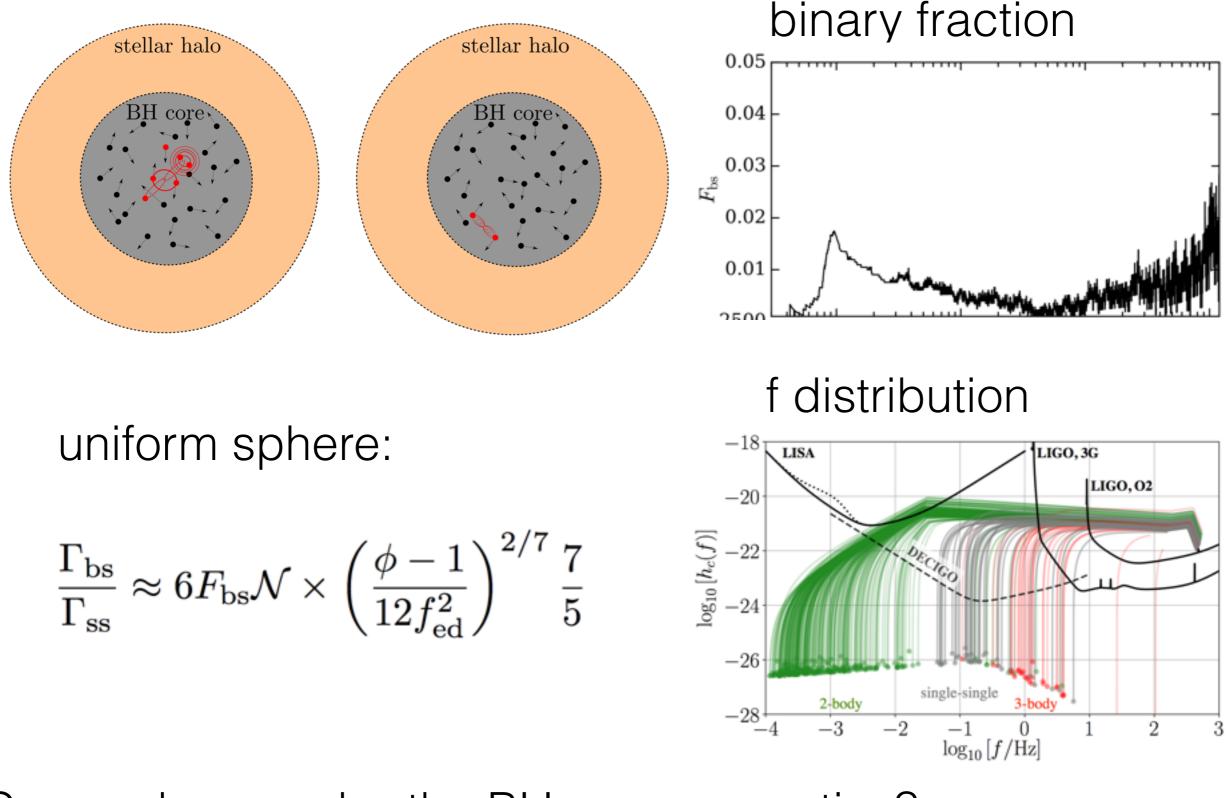
Newtonian codes underestimate the rate by more than a factor of 100!



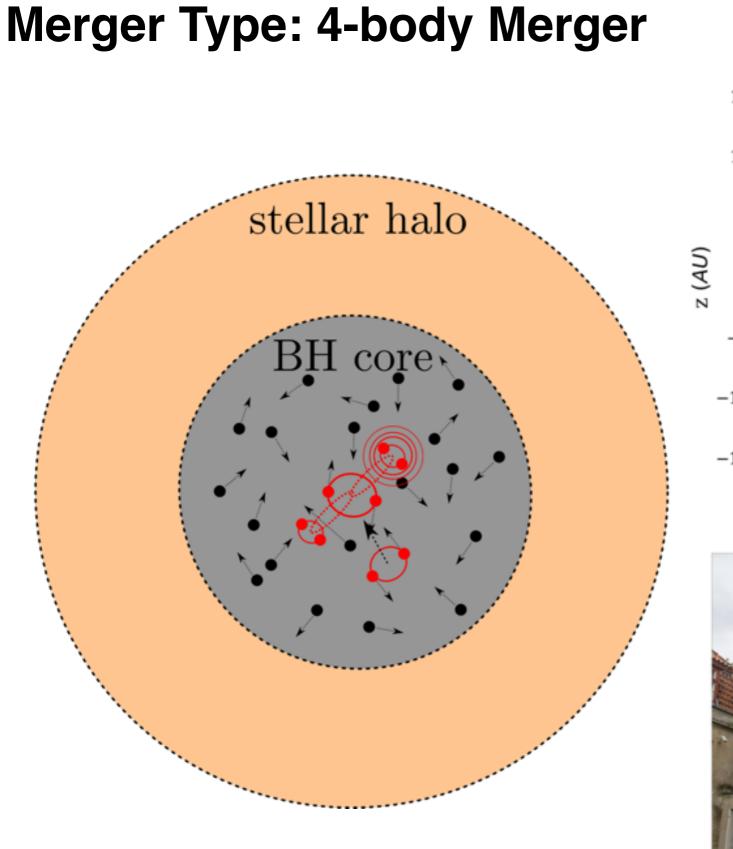
Eccentric LIGO sources

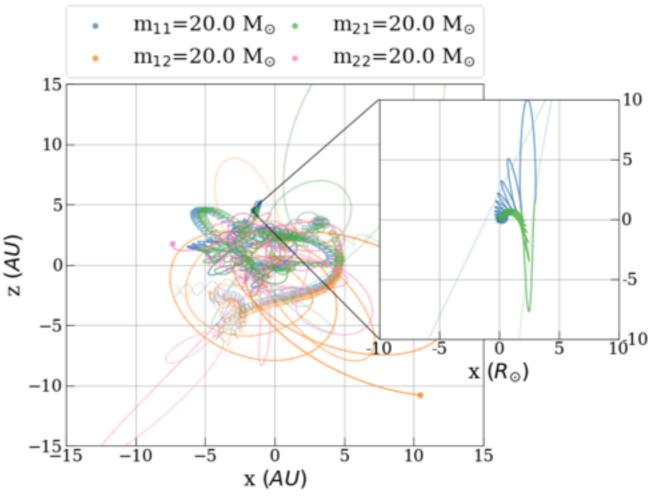


Comparing binary-single and single-single:

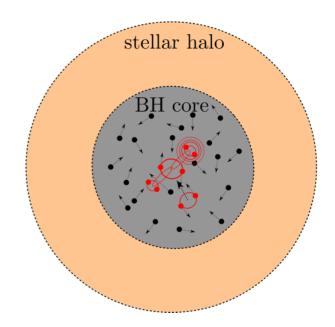


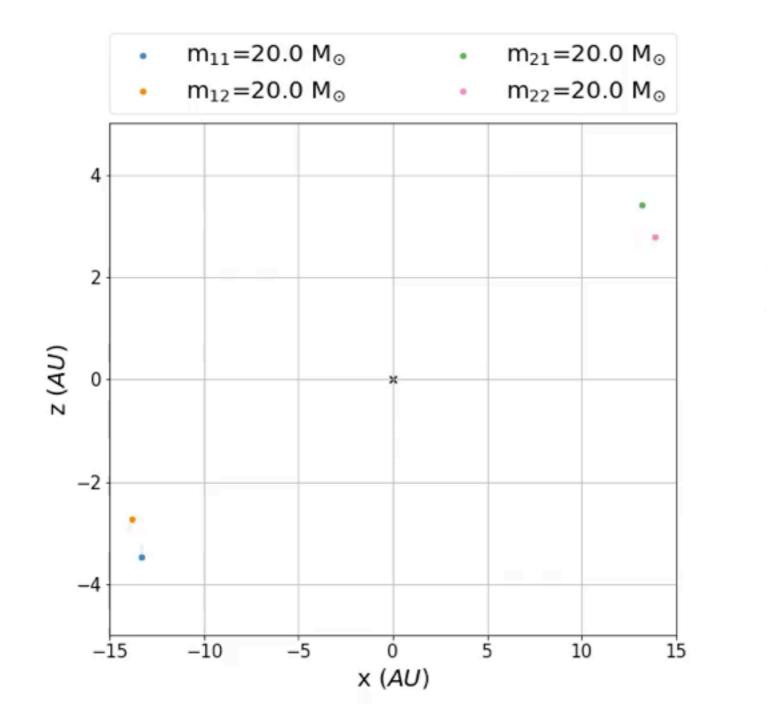
Can we here probe the BH core properties?

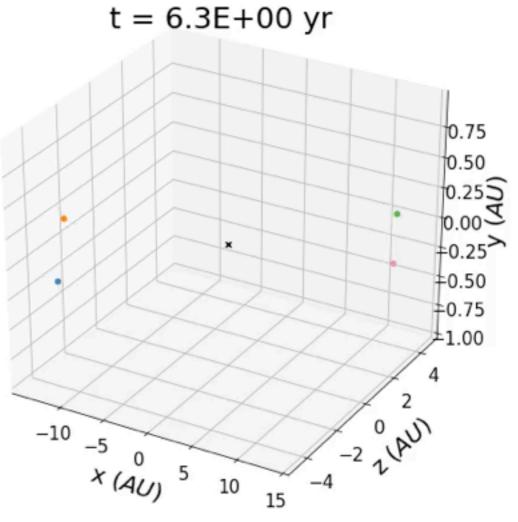


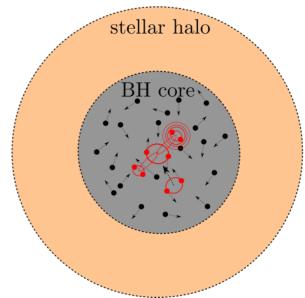


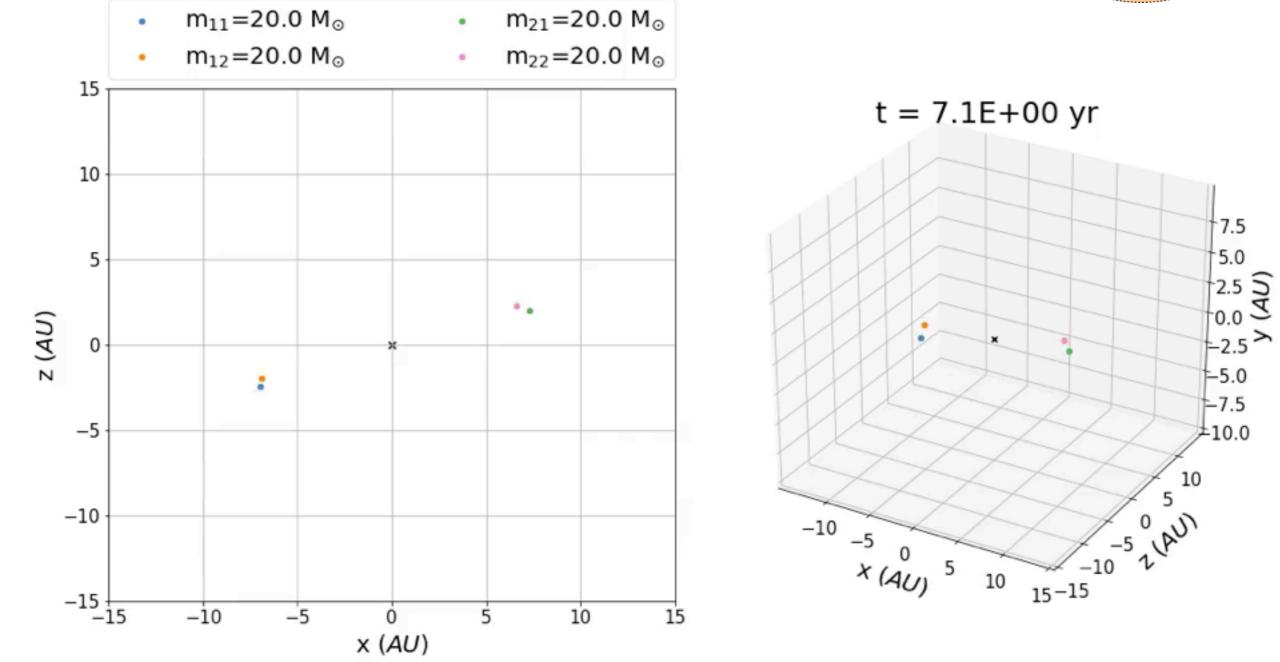


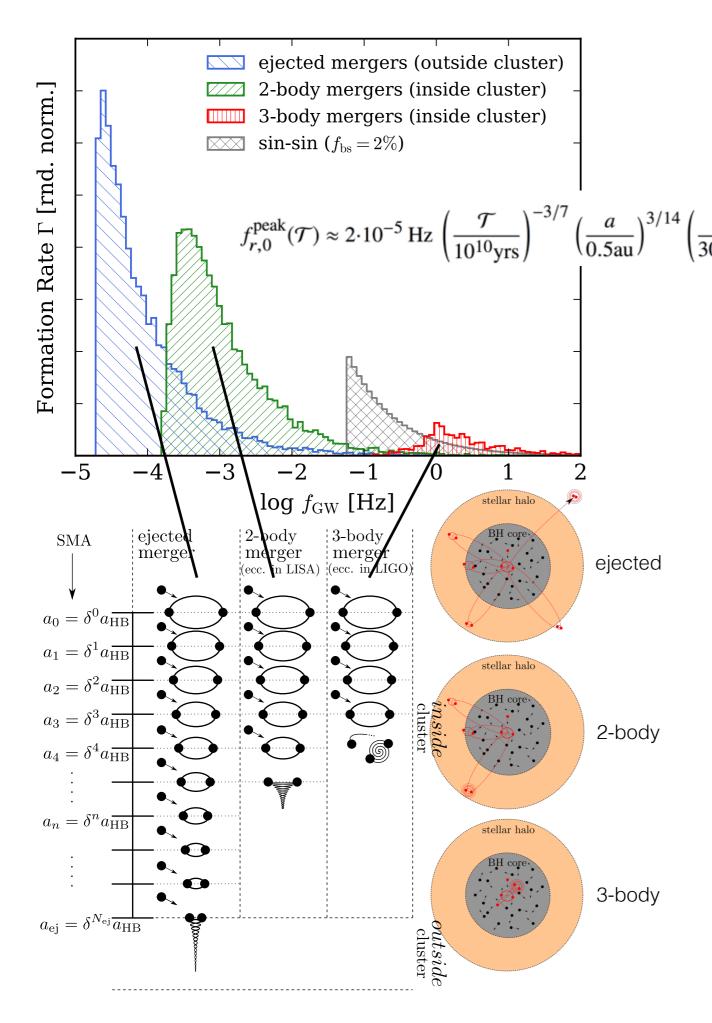










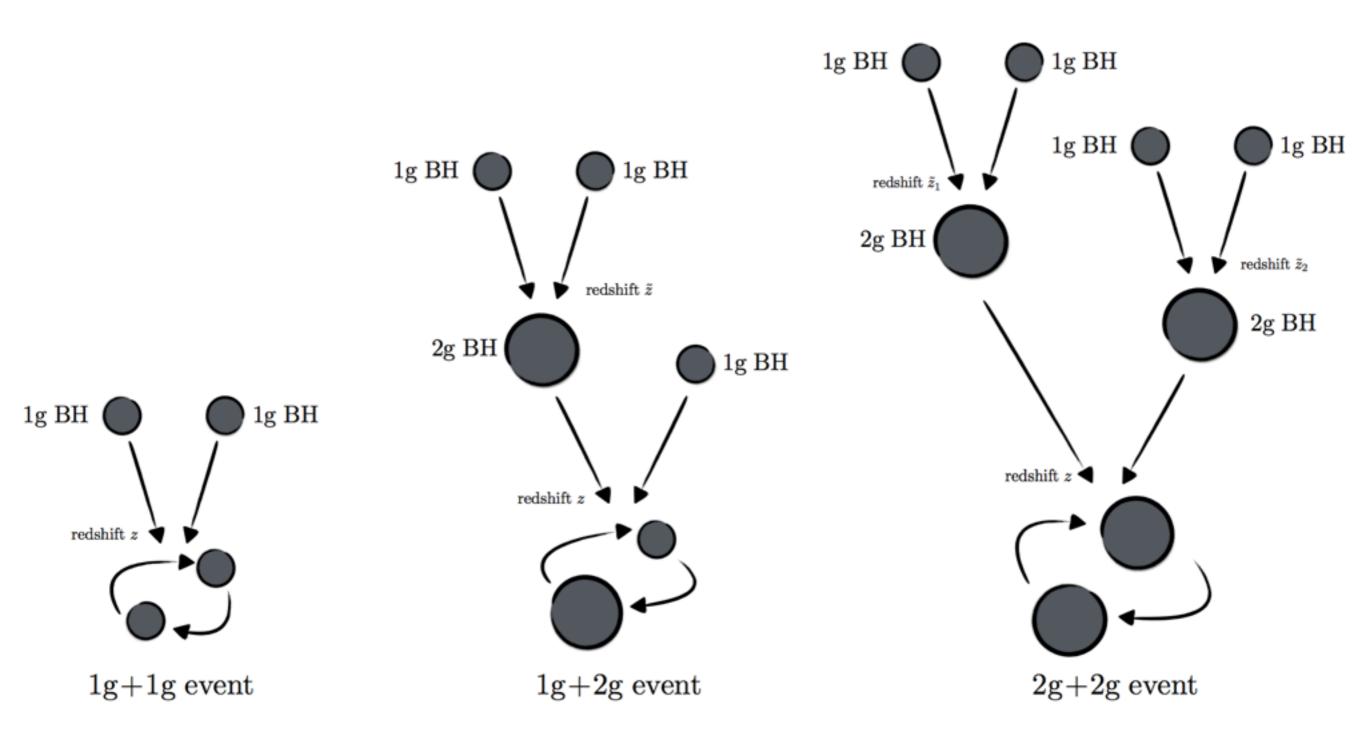


Peak Normalizations:

$$\begin{split} P_{i} &\approx F_{i} \times \left(\frac{\tau_{i}(a_{\rm ej})}{t_{\rm GW}^{e=0}(a_{\rm ej})}\right)^{2/7} \\ F_{\rm in} &\approx (7/10)/(1-\delta) \approx 3 \\ F_{\rm GW} &\approx (7/5)/(1-\delta) \times N_{\rm MS} \approx 120 \end{split}$$
$$\begin{aligned} a_{\rm ej} &\sim 0.5 \text{ AU} \qquad M \sim 30 M_{\odot} \\ \tau_{\rm in} &\sim 10^{7} \text{ years } \tau_{\rm GW} \sim 0.1 \text{ year} \end{aligned}$$
$$\begin{split} P_{\rm in} &\approx 0.15 \quad P_{\rm GW} \approx 0.03 \\ P(t_{\rm GW}(a_{\rm ej}) < T_{\rm H}) \approx 0.35 \end{split}$$

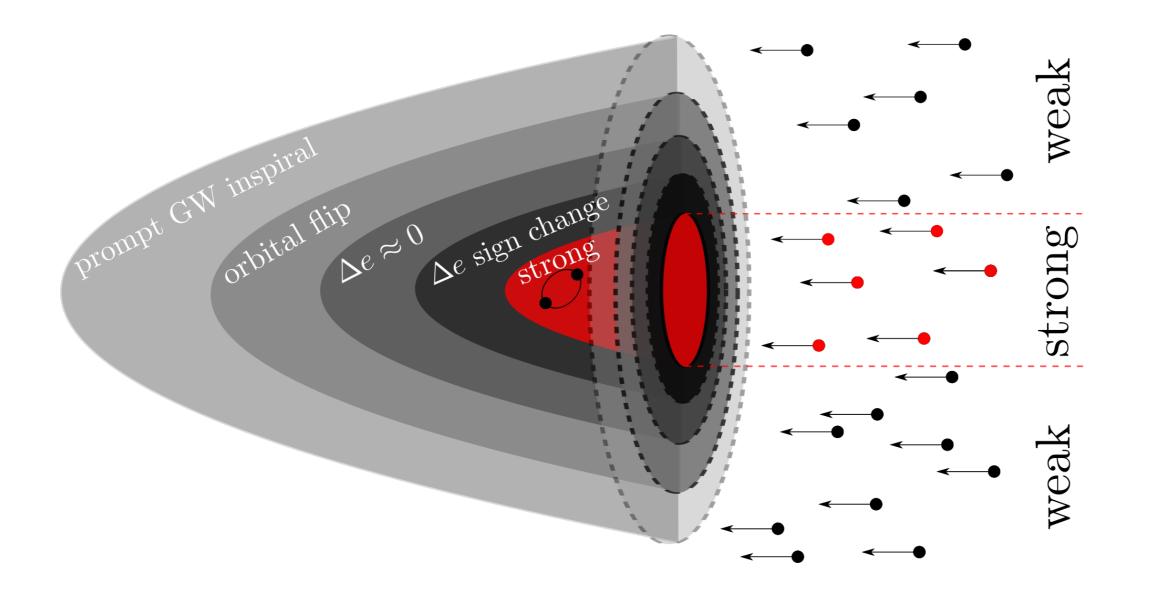
 $0.82\,\times\,0.35\,\,\approx\,0.3$

start to reach 10% high ecc. LIGO mergers



Merger Type: Secular-processes

- work done with Adrian Hamers (IAS)



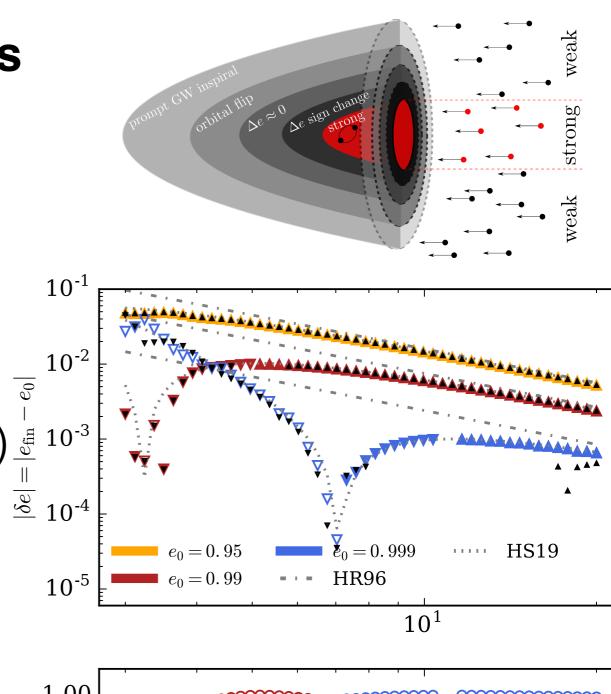
Merger Type: Secular-processes Why is this important?

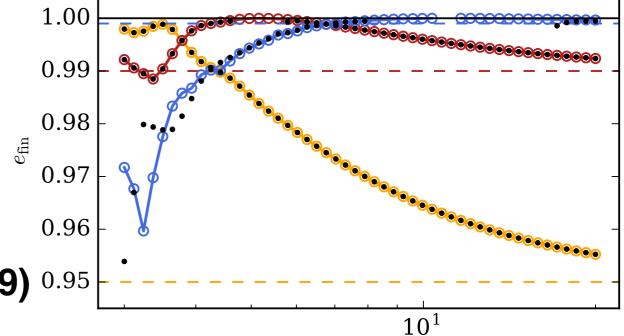
1.order (Heggie, Rasio 96)

$$\delta e = -\frac{15\pi}{16} \left(\frac{2m_3^2 a^3}{M_{123} M_{12} r_p^3} \right)^{1/2} e^{\sqrt{1 - e^2} \sin 2\Omega \sin^2 i}$$

2.order 2x(Hamers, Samsing 19) $\Delta e_{\rm SO} = \Delta e_{\rm FO} + \epsilon^2 \frac{3}{512} \pi e_0 \left[-100 \left(1 - e_0^2 \right) \sin 2\Omega \right] \left\{ (5\cos i + 3\cos 3i)\cos 2\omega + 6\sin i\sin 2i \right\} \\ + 4\cos 2i \left\{ 3\pi \left(81e_0^2 - 56 \right) - 200 \left(1 - e_0^2 \right) \right. \\ \left. \cos 2\Omega \sin 2\omega \right\} + 3\pi \left\{ 200e_0^2 \sin^4 i\cos 4\Omega \right. \\ \left. + 8 \left(16e_0^2 + 9 \right) \sin^2 2i\cos 2\Omega \right. \\ \left. + \left(39e_0^2 + 36 \right)\cos 4i - 299e_0^2 + 124 \right\} \right],$

and 1x(Samsing, Hamers, Types 19) 0.95





non-parabolic limit...

16 Hamers & Samsing

 $g_{e}^{(1)} = \left\{ -\frac{15}{512} \pi \left(2541 e_{\lambda}^{3} e_{y} + e_{\lambda}^{2} \left(36\pi \left(e_{y}^{2} - 163 e_{\xi}^{2} + 55 g_{y}^{2} - 5 g_{\xi}^{2} \right) - 847 j_{x} j_{y} \right) + e_{x} \left(2037 e_{y}^{3} - e_{y} \left(294 e_{\xi}^{2} + 637 g_{\lambda}^{2} + 2992 \pi j_{x} j_{y} + 3969 g_{y}^{2} + 4074 g_{\xi}^{2} - 420 \right) + 24 e_{z,jz} (76 \pi j_{x} - 77 j_{y}) \right) \\ = 60 \pi f_{x}^{2} \left(3e_{y}^{2} + 15e_{\xi}^{2} - 3j_{y}^{2} - g_{\xi}^{2} \right) + 7 j_{x} \left(j_{y} \left(163 e_{y}^{2} - 482 e_{\xi}^{2} - 62 g_{\xi}^{2} + 20 \right) - 264 e_{y} e_{z,jz} + g_{y}^{3} \right) \right)$

 $+12\pi \left(3e_y^4 + e_y^2 \left(-91e_z^2 - 36j_y^2 + 3j_z^2 + 4\right) + 72e_ye_zj_yj_z + 96e_z^4 + e_z^2 \left(81j_y^2 + 32j_z^2 - 12\right) - 15j_y^4 + 15j_y^2j_z^2 + 12j_y^2 - 4j_z^2\right) - 49j_X^3j_y\right)$

 $\frac{15}{512}\pi \left(2541e_{X}^{4}+36\pi e_{X}^{3}e_{Y}+e_{X}^{2}\left(2037e_{Y}^{2}-2513e_{Z}^{2}-1484j_{X}^{2}+1380\pi j_{X}j_{Y}-903j_{Y}^{2}-7623j_{Z}^{2}+420\right)\right)$

 $+ 4e_x \left(9\pi e_y^3 + e_y \left(3\pi \left(398e_z^2 - 35j_y^2 + 18j_z^2 + 4\right) - 453\pi j_x^2 - 658j_x J_y\right) - 2e_{zJz} (469j_x + 36\pi J_y)\right) - e_z^2 \left(707e_y^2 + 3213j_x^2 - 72\pi j_x J_y + 7 \left(j_y^2 - 480j_z^2 + 20\right)\right) + 707e_y^2 J_x^2 - 12\pi e_y^2 J_x J_y - 2037e_y^2 J_z^2 + 56e_y e_{zJz} (47j_y - 12\pi j_x) + 1120e_z^4 - 49j_x^4 - 180\pi j_x^3 J_y + 7j_x^2 J_y^2 + 637j_x^2 J_z^2 + 140j_x^2 - 180\pi j_x J_y^2 + 120\pi j_x J_y J_z^2 + 144\pi j_x J_y + 903j_y^2 J_z^2 - 420j_z^2\right), \\ \frac{15}{512} \pi \left(2892\pi e_x^3 e_z - e_y \left(e_z \left(-2219e_x^2 + 1561j_x^2 + 264\pi j_x J_y + 3099j_y^2 - 140\right) + 2e_x J_z (2891j_x + 780\pi j_y) + 1120e_z^2\right) + 3e_x^2 J_z (3143j_y - 492\pi j_x) + e_y^2 (2892\pi e_x e_z + 84\pi j_x J_z + 2471j_y J_z) \right) \\ - 2e_x e_z \left(6\pi \left(224e_z^2 + 179j_y^2 - 28\right) + 1206\pi j_x^2 - 679j_x J_y\right) + 707e_y^3 e_z + 3j_z \left(-7j_y \left(160e_z^2 + 51j_x^2 - 20\right) + 4\pi j_x \left(96e_z^2 + 35j_x^2 - 12\right) + 140\pi j_x J_y^2 - 301j_y^3\right)\right)\right) \right\};$

 $\mathbf{g}_{e}^{(2)} = \left\{ \frac{225}{12568} \pi \left(13041 e_{y}^{5} + 288 e_{x} \pi e_{y}^{4} + \left(11626 e_{x}^{2} + 36082 e_{z}^{2} + 2458 y_{x}^{2} - 23654 y_{y}^{2} - 52164 y_{z}^{2} + 640 y_{x} y_{y} \pi - 672 \right) e_{y}^{3} \right\}$

 $+ 6 \left(72\pi e_X^3 + \left(820\pi j_X^2 + 1583 j_Y j_X + 8 \left(-527e_Z^2 - 234 j_Y^2 + 9j_Z^2 + 12 \right) \pi \right) e_X + 2e_Z j_Z (200\pi j_X + 11097 j_Y) \right) e_Y^2$

 $+ \left(20097e_{A}^{4} + 2\left(12302e_{Z}^{2} - 507j_{A}^{2} - 32777j_{Y}^{2} - 30636j_{Z}^{2} - 13472j_{A}j_{Y}\pi + 2016\right)e_{A}^{2} + 48e_{Z}j_{Z}(53j_{A} + 216j_{Y}\pi)e_{A} + 68096e_{Z}^{4} + 3825j_{A}^{4} + 1925j_{Y}^{4} - 3136j_{A}^{2} - 1570j_{A}^{2}j_{Y}^{2} + 1568j_{Y}^{2} - 4916j_{A}^{2}j_{Y}^{2} - 14124j_{Y}^{2}j_{Z}^{2} + 1344j_{Z}^{2}\right)e_{Y} \\ + \left(4e_{Z}^{2}\left(8007j_{A}^{2} + 600j_{Y}A_{J}A - 19009j_{Y}^{2} - 2688j_{Z}^{2} - 3218\right) + 3200j_{X}J_{Y}^{2}\pi - 960j_{X}j_{Y}j_{Y}^{2}\pi - 1200j_{X}j_{Y}x^{2} + 8560\right)e_{Y} + 8e_{Z}^{2}e_{Z}j_{Z}(4172a_{Z}A + 1167j_{Y})e_{Y} + 8e_{Z}^{2}e_{Z}j_{Z}(4172a_{Z}A + 1167j_{Y})e_{Z} + 8e_{Z}^{2}e_{Z}j_{Z}(4172a_{Z}A + 1167j_{Z})e_{Z} + 8e_{Z}^{2}e_{Z}j_{Z}(4172a_{Z}A + 1167j_{Z})e_{Z} + 8e_{Z}^{2}e_{Z}j_{Z} + 8e_{Z}^{2}e_{Z}j_{Z} + 8e_{Z}^{2}e_{Z}j$

 $-8e_{\chi J\chi}\left(-3803f_y^3+1000f_X\pi f_y^2+\left(-17026e_{\chi}^2-5767f_X^2+2128\right)f_y+40f_X\left(32e_{\chi}^2+15f_X^2-4\right)\pi\right)-4e_X^3\left(7343f_Xf_y+216\left(73e_{\chi}^2-19f_y^2+f_{\chi}^2\right)\pi\right)$

 $+ \delta e_{N} \left(281j_{Y} j_{N}^{2} + \delta 0 \left(115e_{Z}^{2} - 17j_{Y}^{2} - 21j_{Z}^{2} \right) \pi j_{N}^{2} + j_{Y} \left(-20602e_{Z}^{2} + 2237j_{Y}^{2} - 3802j_{Z}^{2} + 672 \right) j_{N} + 8 \left(86de_{Z}^{4} + 3 \left(203j_{Y}^{2} + 32j_{Z}^{2} - 36 \right) e_{Z}^{2} - 130j_{Y}^{4} - 12j_{Z}^{2} + j_{Y}^{2} \left(65j_{Z}^{2} + 108 \right) \right) \pi \right) \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 60 \left(115e_{Z}^{2} - 17j_{Y}^{2} - 21j_{Z}^{2} \right) + j_{Y}^{2} \left(65j_{Z}^{2} + 108 \right) \right) \pi \right) \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} - 36 \right) e_{Z}^{2} + 102j_{Z}^{2} + j_{Y}^{2} \left(65j_{Z}^{2} + 108 \right) \right) \pi \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} - 36 \right) e_{Z}^{2} + 22j_{Z}^{2} + 2j_{Z}^{2} \left(65j_{Z}^{2} + 108 \right) \right) \pi \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} - 36 \right) e_{Z}^{2} + 2j_{Z}^{2} \left(65j_{Z}^{2} + 108 \right) \right) \pi \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} - 36 \right) e_{Z}^{2} + 2j_{Z}^{2} \left(65j_{Z}^{2} + 108 \right) \right) \pi \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} + 32j_{Z}^{2} + 32j_{Z}^{2} + 32j_{Z}^{2} \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} + 32j_{Z}^{2} + 32j_{Z}^{2} + 32j_{Z}^{2} \right) + \delta e_{N} \left(281j_{Y} j_{N}^{2} + 32j_{Z}^{2} + 32j$

 $-\frac{223}{32768}\pi\left(20097e_{A}^{5}+432e_{Y}\pi e_{A}^{4}+\left(31626e_{Y}^{2}-2884e_{E}^{2}-28398j_{A}^{2}-8294j_{Y}^{2}-80388j_{E}^{2}+8896j_{X}j_{Y}\pi+4052\right)e_{A}^{3}\right)$

 $+ 4 \left(146 \pi \sigma_y^2 + \left(-3896 \pi J_R^2 - 10513 J_{y,J,K} + 8 \left(1363 \sigma_z^2 - 170 J_y^2 + 27 J_z^2 + 24 \right) \pi \right) \sigma_y + 2 \sigma_z J_z (2897 J_R + 472 J_y \pi) \right) \sigma_X^2$

 $+ \left(13041w_y^4 + 2\left(7126w_L^2 - 3309j_R^2 + 3551j_Y^2 - 31628j_L^2 - 64j_Rj_Y\pi - 336\right)w_Y^2 + 16w_Zj_Z(8207j_Y - 1256j_R\pi)w_Y + 57346w_L^4 + 4949j_R^4 + 2145j_Y^4 - 448j_R^2\right)w_R$

 $+ \left(14558_{f_{R}^{2}J_{P}^{2}} - 3264_{f_{R}^{2}}J_{P}^{2} - 3228_{f_{R}^{2}J_{P}^{2}}J_{P}^{2} - 8664_{f_{R}^{2}}J_{P}^{2} - 4e_{q}^{2}\left(10699_{f_{R}^{2}} - 4272_{Jy}\pi_{Jx} - 2601_{f_{R}^{2}}J_{P}^{2} - 2812\right) - 1930_{f_{R}J_{P}^{2}}\pi - 2880_{JxJ_{P}J_{P}^{2}}\pi - 320J_{R}^{3}J_{P}\pi + 1516f_{R,Jy}\pi + 500\right)e_{R} + 4\left(26\pi e_{y}^{2} - \left(266\pi J_{R}^{2} + 9776y_{JyJ_{R}} + 8\left(81e_{z}^{2} - 5J_{p}^{2} + 8j_{z}^{2} - 12\right)\pi\right)e_{y}^{2} + 6e_{zJ_{z}}(1613_{Jx} + 312_{Jy}\pi)e_{y}^{2}\right)$

 $+4\left[\left(300\pi g_{R}^{4}-170f_{Jy}g_{R}^{3}+60\left(15e_{2}^{2}+30g_{2}^{2}+30g_{2}^{2}-8\right)\pi g_{R}^{2}-g_{y}\left(20602e_{1}^{2}+50g_{2}^{2}+3802g_{2}^{2}-1008\right)g_{R}+4\left(706e_{2}^{4}+6\left(13f_{y}g_{2}^{2}+32g_{1}^{2}-36\right)e_{2}^{2}+125g_{2}^{4}-24g_{2}^{2}-16g_{2}^{2}\left(7g_{2}^{2}+12\right)+16\right)\pi\right)e_{y}\right)\\ +4\left(2e_{2}g_{2}\left(4218g_{R}^{3}-40g_{y}\pi g_{R}^{2}+\left(14356e_{2}^{2}+2407g_{y}^{2}-1702\right)g_{R}+88g_{y}\left(32e_{2}^{2}+5g_{y}^{2}-4\right)\pi\right)\right)\right),$

 $-\frac{225}{2048\pi}\left[\left(1624\pi e_{1}^{2}+32\left(20\pi f_{2}^{2}+21f_{2}f_{3}+4\left(e_{2}^{2}-11f_{2}^{2}-2\right)\pi\right)e_{1}^{2}\right)e_{2}^{2}\right] + 0f_{2}f_{3}^{2}+24f_{2}g_{3}\pi e_{1}^{2}+(15\pi f_{3}^{2}+319)g_{2}f_{3}^{2}-10\left(12e_{2}^{2}+17f_{2}^{2}-8\right)\pi e_{1}^{2}+373f_{2}^{2}-20g_{1}^{2}g_{2}^{2}-30g_{1}^$

$$\begin{split} \mathbf{h}_{\mathbf{e}}^{(0)} &= \left[\frac{3}{212}\pi \left(-1200\pi e_{A}^{3} f_{z} + 15e_{X}^{2} \left(1361e_{Y} f_{z} + 80\pi e_{Z} f_{X} + \left(569 + 384\pi^{2}\right)e_{Z} f_{Y}\right) - 6e_{X} \left(300\pi e_{Y}^{2} f_{z} + 15e_{Y} e_{Z} \left(\left(64\pi^{2} - 109\right) f_{X} + 160\pi f_{Y}\right) + f_{Z} \left(120\pi f_{X}^{2} + \left(581 + 192\pi^{2}\right) f_{X} f_{Y} + 20\pi f_{Y}^{2}\right)\right) - 7573e_{Y}^{2} f_{Z} + 15e_{Y} e_{Z} \left(164\pi f_{Y} + 681f_{Y}\right) + e_{Y} f_{Z} \left(\left(1152\pi^{2} - 3353\right) f_{X}^{2} + 960\pi f_{H} g_{Y} + 29\pi f_{Y}^{2}\right) + e_{Z} \left(720\pi f_{X}^{2} - 313f_{X}^{2} f_{Y} + 1060\pi f_{H} g_{Y}^{2} - 927f_{Y}^{2}\right)\right) - \frac{9}{252\pi} \pi \left(-4475e_{X}^{2} f_{Z} - 5e_{Y}^{2} (60\pi e_{Y} f_{Z} + 10e_{Y} g_{Z} + 1$$

 $+e_Y\left(-i30e_Xe_Zj_2+i5\left(21+16\pi^2\right)e_Z^2j_X+j_Z^2\left(68\pi^2j_X+7j_X-240\pi_fy\right)\right)+15e_Xe_Z^2\left(80\pi_fX+\left(133-48\pi^2\right)f_Y\right)+e_Xj_Z^2\left(\left(77-48\pi^2\right)f_Y-480\pi_fX\right)+1300\pi e_Y^2e_Zj_Z-12e_Zj_Xj_Z(20\pi_fX+6)f_Y)\right)\right)+15e_Xe_Z^2\left(66\pi_fX+\left(133-48\pi^2\right)f_Y\right)+126\pi_fX+2$

The functions associated with the vector angular-momentum changes are given by

 $f_{J}^{(0)} = \left\{-\frac{3}{2}\pi(5e_{y}e_{z} - J_{y}J_{z}), \frac{3}{2}\pi(5e_{x}e_{z} - J_{x}J_{z}), 0\right\};$

 $f_{j}^{(1)} = \left[-\frac{75}{16} \pi (-7e_{x}e_{y}e_{z} + e_{xJyJz} + e_{yJxJz} + e_{zJxJy}), \frac{15}{32} \pi \left(e_{z} \left(-73e_{x}^{2} - 3e_{y}^{2} + 15j_{x}^{2} + 5j_{y}^{2} - 4 \right) + 10j_{z} (3e_{xJx} + e_{yJy}) + 32e_{z}^{3} \right), \frac{15}{32} \pi \left(e_{y} \left(3e_{x}^{2} - 32e_{z}^{2} - 5j_{x}^{2} - 15j_{y}^{2} + 4 \right) - 10e_{xJxJy} + 3e_{y}^{3} \right) \right];$

 $g_{J}^{(0)} = \left\{ \frac{3}{16} \pi \left(75 e_{XJY}^2 + 60 \pi e_X e_Y J_Y + J_X \left(5J_X J_Y - 6\pi \left(10 e_Y^2 + J_Z^2 \right) \right) - 50 e_Y e_Z J_Z + 10 e_Z^2 (5J_Y - 9\pi J_X) \right),$

 $-\frac{3}{16}\pi\left(15e_{A}^{2}(5j_{A}+4\pi j_{Y})-10e_{A}(6\pi e_{Y}j_{A}+5e_{Y}j_{Y}+15e_{Z}j_{Z})+50e_{Y}^{2}j_{A}+90\pi e_{Z}^{2}j_{Y}+5j_{A}^{3}-10j_{X}j_{Z}^{2}+6\pi j_{Y}j_{Z}^{2}\right)$

 $-\frac{15}{8}\pi(5e_xe_yj_{\xi}+5e_xe_{\xi}j_y+5e_ye_{\xi}j_x+j_xj_yj_{\xi});$

 $g_{J}^{(1)} = \left\{ \frac{15}{512} \pi \left(-2541 e_{x}^{3} J_{y} + e_{y} \left(7 J_{x} \left(121 e_{x}^{2} + 682 e_{z}^{2} - 63 J_{y}^{2} + 62 J_{z}^{2} - 20 \right) - 24 \left(116 \pi e_{x}^{2} J_{y} - 322 e_{x} e_{z} J_{z} + \pi J_{y} \left(41 e_{z}^{2} - 5 J_{y}^{2} - 5 J_{z}^{2} + 8 \right) \right) = 49 J_{x}^{3} \right) - 96 \pi e_{x}^{2} e_{z} J_{z} \\ + 3 e_{y}^{2} (1080 \pi e_{x} J_{x} + 343 e_{x} J_{y} - 32 \pi e_{z} J_{z}) + e_{x} \left(7 J_{y} \left(42 e_{z}^{2} + 91 J_{x}^{2} - 258 J_{z}^{2} - 60 \right) + 360 \pi J_{x} \left(11 e_{z}^{2} + J_{z}^{2} \right) + 120 \pi J_{x} J_{y}^{2} + 93 J_{y}^{3} \right) + e_{y}^{3} (456 \pi J_{y} - 707 J_{x}) \\ - 64 e_{z} J_{z} \left(3 \pi \left(8 e_{z}^{2} - 1 \right) + 15 \pi J_{x}^{2} - 7 J_{x} J_{y} \right) \right).$

 $\frac{15}{256}\pi\left(e_{A}^{3}(847j_{A}+1446\pi j_{Y})-e_{A}^{2}(le_{Y}(558\pi j_{A}+511j_{Y})+6748e_{z,f_{z}})+e_{x}\left(7j_{x}\left(196e_{Y}^{2}-359e_{c}^{2}-64j_{Y}^{2}+91j_{c}^{2}+40\right)-6\pi j_{y}\left(29e_{Y}^{2}-406e_{c}^{2}+35j_{Y}^{2}-50j_{c}^{2}-28\right)-343j_{A}^{3}-270\pi j_{A}^{2}j_{Y}\right)-54\pi e_{Y}^{3}j_{A}-1372e_{Y}^{2}e_{z,f_{z}}+e_{Y}\left(3e_{c}^{2}(12\pi j_{X}+511j_{Y})+210\pi j_{A}^{3}+217j_{c}^{2}j_{Y}+6\pi j_{x}\left(25j_{Y}^{2}-30j_{c}^{2}-12\right)-217j_{Y}j_{c}^{2}\right)+8e_{z,f_{z}}\left(280e_{c}^{2}+49j_{A}^{2}-60\pi j_{x,Y}+56j_{Y}^{2}-39\right)\right),$

 $\frac{15}{512}\pi\left(36\pi e_X^3 f_c + e_X^2(3549 e_Y f_c - 492\pi e_z f_X + 2219 e_z f_Y) + 2e_X\left(18\pi e_Y^2 f_c + e_Y e_c(60\pi f_Y - 21f_X) + f_c\left(-6\pi\left(32e_z^2 + 25f_y^2 - 4\right) + 30\pi f_X^2 + 49f_X f_Y\right)\right) + e_z\left(-7f_Y\left(337e_y^2 + 23f_X^2 - 20\right) - 36\pi f_X\left(17e_y^2 + 5f_X^2 - 4\right) - 180\pi f_X f_y^2 + 7f_y^3\right) + e_yf_c\left(2037e_y^2 - 1071f_X^2 + 360\pi f_X f_Y - 469f_y^2 + 420\right) - 3360e_Ye_y^2 f_z - 32e_y^3(36\pi f_X + 35f_Y)\right)\right)$

2 Hamers & Samsing

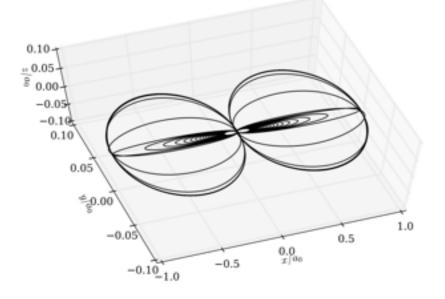
Terms of order	Number of terms in Δe	
	E = 1	E > 1
$\epsilon_{\rm SA}$ (all)	16	60
$\epsilon_{\rm SA}$	2	8
$\epsilon_{\rm SA}\epsilon_{\rm oct}$	14	52
$\epsilon_{\rm SA}^2$ (all)	193	55,895
ϵ_{SA}^2	17	1,871
$\epsilon_{SA}^2 \epsilon_{oct}$	60	16,035
$\epsilon_{SA}^{2} \epsilon_{SA}^{2} \epsilon_{oct}^{2} \epsilon_{SA}^{2} \epsilon_{oct}^{2}$	116	37,989
$\epsilon_{\rm SA}^3$ (all)	1,146	2,931,541
ϵ_{SA}^3	54	38,366
$\epsilon_{SA}^{3} \epsilon_{oct}$ $\epsilon_{SA}^{3} \epsilon_{oct}^{2}$ $\epsilon_{SA}^{3} \epsilon_{oct}^{2}$	175	289,496
$\epsilon_{sA}^3 \epsilon_{oct}^2$	311	856,072
$\epsilon_{SA}^{3} \epsilon_{oct}^{3}$	606	1,747,607

Merger Type: Secular-processes

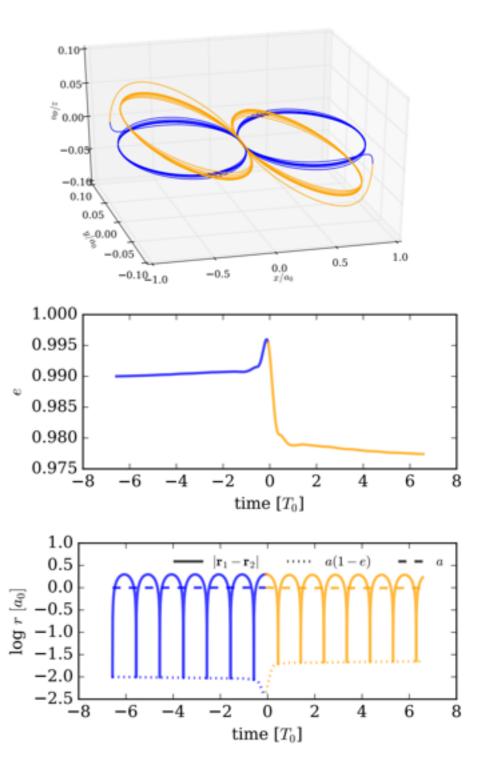
PN effects?

2.5

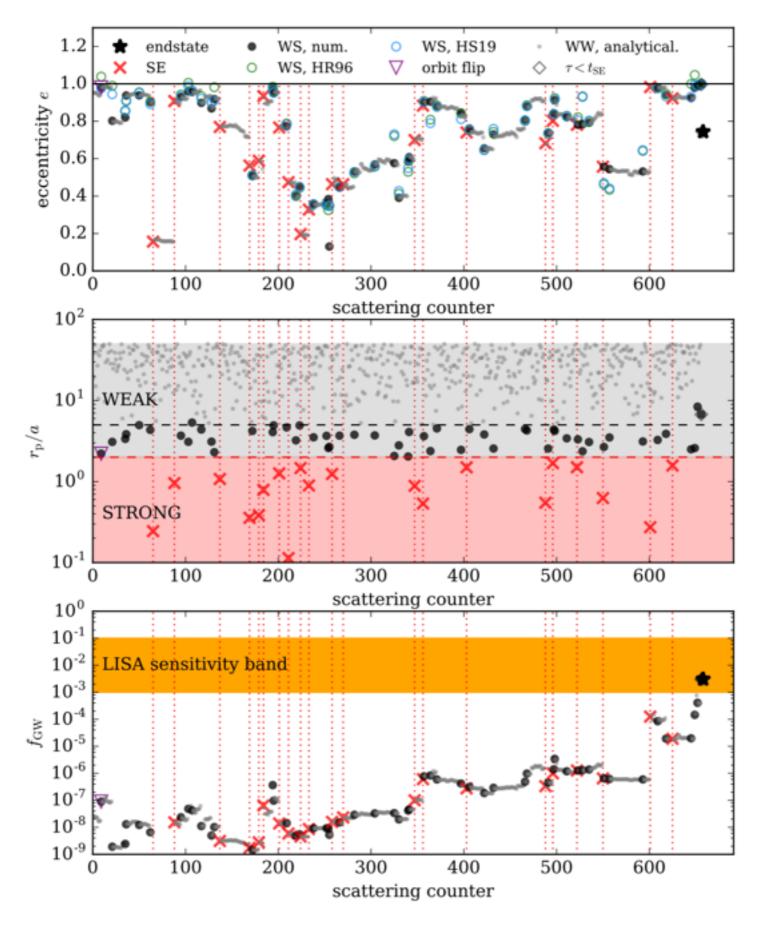
1,2



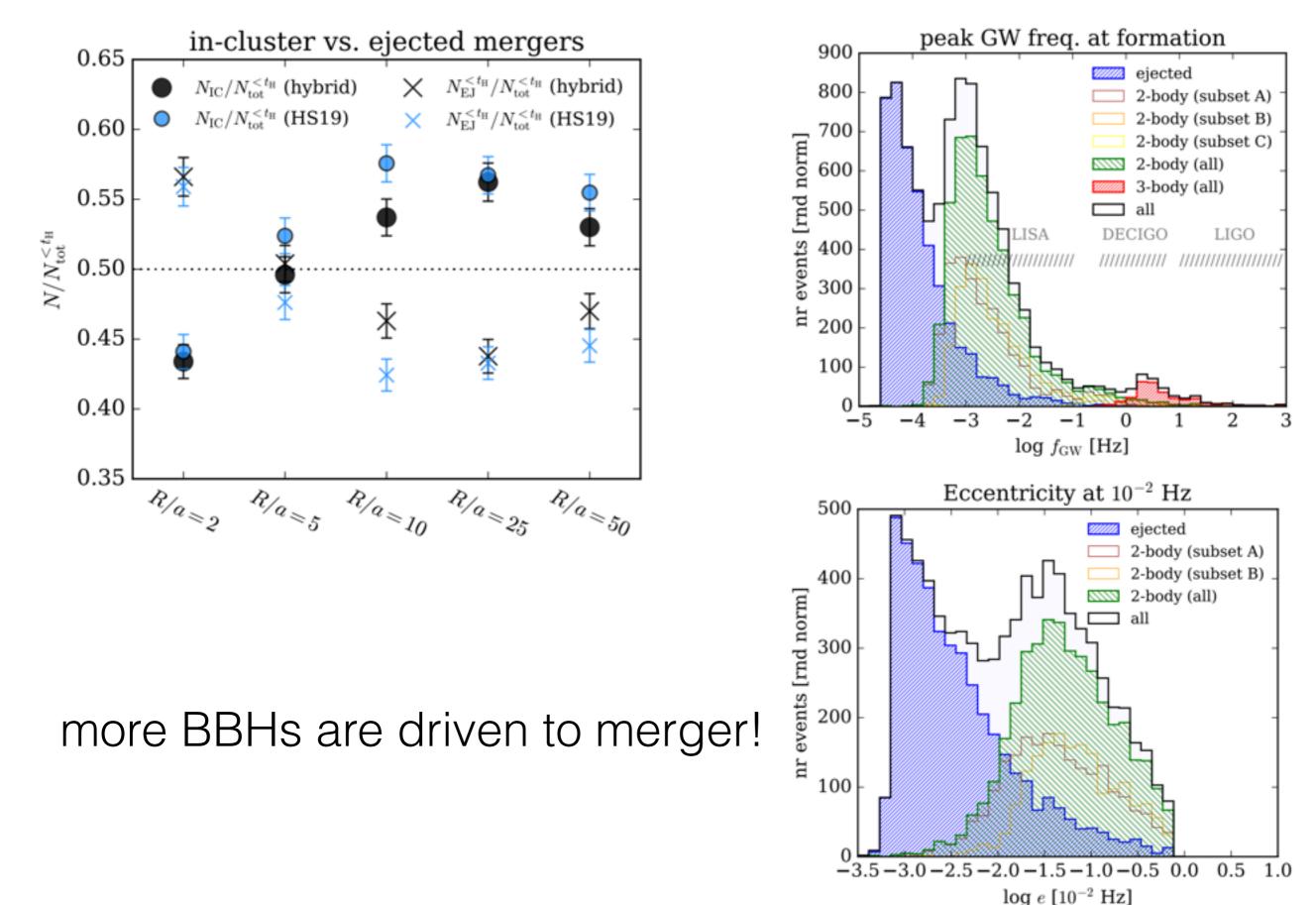
$$\begin{split} \frac{r_{\rm p}}{a_0} \gtrsim \frac{a_0^{2/3}}{\mathscr{R}_{\rm m}^{2/3}} (1 - e_0^2)^{2/3} \\ \gtrsim 10^3 \times \left(\frac{a_0}{0.5 {\rm AU}}\right)^{2/3} \left(\frac{m}{20 M_{\odot}}\right)^{-2/3} \left(1 - \left(e_0/0.99\right)^2\right)^{2/3} \end{split}$$



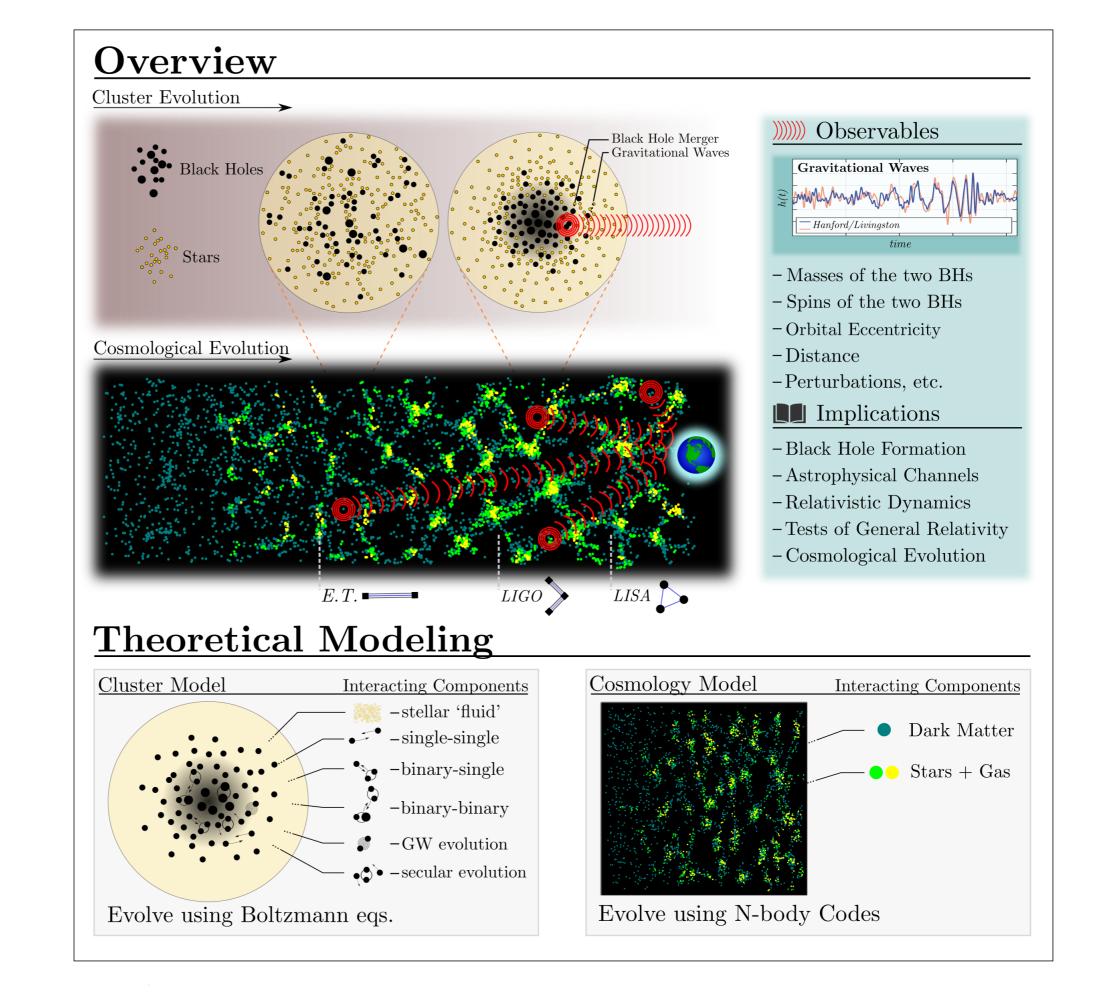
Merger Type: Secular-processes



Results from our MC code



Future



THANK YOU